

Pressure in hot QCD – how non-perturbative is it?

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- “QCD is perturbative when $T, \mu \gg \Lambda_{QCD}$ ”

⇒ Weakly interacting gas of quarks and gluons, *quark-gluon plasma*

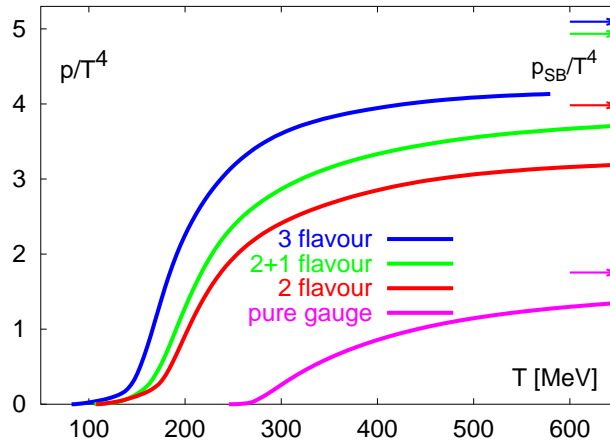
- Can apply perturbation theory
- But: *infrared singularities* ⇒ perturbation theory breaks down at finite order!

Linde 1980

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- Pressure $p = -F/V$ at $T \gtrsim 4T_c$
 - This is where standard lattice QCD data stops
 - Use *3d effective theory*
 - explicit **Lat** ↔ **\overline{MS}** connection
 - use P.T. as much as you can (all calculable contributions)
 - rest non-perturbatively
 - Heavy-ion collisions, Cosmology, ...
 - Resummation techniques
 - Andersen, Braaten, Petitgirard, Strickland;
 - Blaizot, Iancu, Rebhan;
 - Parwani, Peshier, Cvetič, Kőgerler ...

Lattice QCD

Standard lattice QCD works very well when $T \lesssim 5-10T_c$:



Karsch 2001

- Pure gauge $N_f = 0$: $p(T \leq T_c) \approx 0$ since glueballs heavy
- $N_f = 2, 3$: at $T < T_c$ gas of (light) pions
- $p'(T_c)$ discontinuous \Rightarrow genuine 1st order phase transition (\exists for pure gauge QCD)

Lattice QCD runs out of steam at $T \sim 5-10T_c!$

- Free energy on a lattice

$$e^{-fV/T} = \int [dU] e^{-S[U]}$$

is finite and (in principle) calculable.

- However, it is UV divergent (“cosmological constant”).
- Solution: subtract f from $T = 0$ lattice

$$p_{\text{phys.}}(T) = -\lim_{a \rightarrow 0} [f_{\text{lat}}(T, a) - f_{\text{lat}}(T = 0, a)]$$

- Lattice spacing $a \ll 1/T$
- $T = 0$ means size of the lattice $L \gg 1/T_c \sim 1/\Lambda_{\text{QCD}}$
- Wide range of scales: $a \ll 1/T \ll 1/T_c \ll L$, all included in $T = 0$ lattice! Very expensive at high T .

- Still a substantial difference to the free Stefan-Boltzmann behaviour.
- What is the relation to perturbation theory?

Pressure in perturbation theory

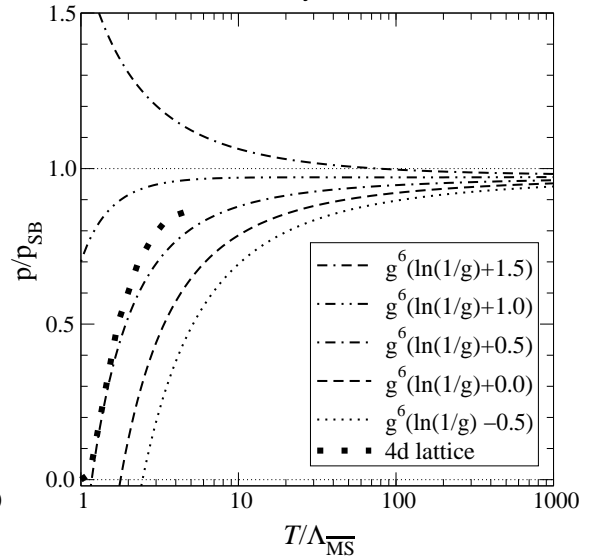
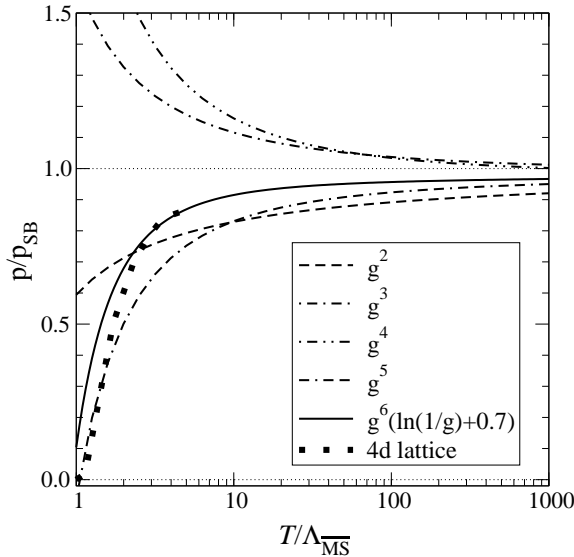
The perturbative expression for the pressure is now fully known:

$p/p_{\text{SB}} = 1$	Stefan-Boltzmann	
$+g^2$	2-loop	(Shuryak 78)
$+g^3$	resummed 2-loop	(Kapusta 79)
$+g^4 \ln 1/g$	resummed 2-loop	(Toimela 83)
$+g^4$	resum 3-loop	(Arnold, Zhai 94)
$+g^5$	resum 3-loop	(Kastening, Zhai 95)
$+g^6 \ln 1/g$	resum 4-loop	(Laine, Kajantie, K.R., Schroeder 02)
$+g^6$	not computable in P.T!	(Linde 80)
$+g^7 \dots$...	

$$p_{\text{SB}} = \frac{\pi^2 T^4}{45} \left(8 - \frac{21 N_f}{4} \right)$$

However, P.T. alone does not provide the answer:

Laine, Kajantie, K.R, Schroeder 02



- Generally bad convergence
- Non-perturbative effects [parametrized here with $g^6 \times (const.)$] can be very significant!

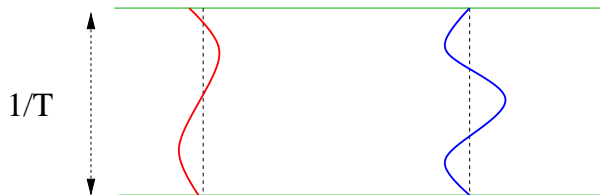
Why is $g^6 T^4$ non-perturbative?

- 1) Finite T ensemble: euclidean metric with imaginary time extent $1/T$, with (anti)periodic b.c for bosons (fermions) \Rightarrow

$$\frac{1}{p^2} \longrightarrow \frac{1}{\bar{p}^2 + \omega_n^2}, \quad \omega_n = \begin{cases} 2n\pi T & n \in \mathbb{Z} \text{ Bosons} \\ (2n+1)\pi T & n \in \mathbb{Z} \text{ Fermions} \end{cases}$$

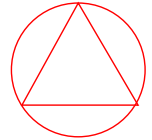
$$\int \frac{d^4 p}{(2\pi)^4} \longrightarrow T \sum_n \int \frac{d^3 p}{(2\pi)^3}$$

Thus, all $n \neq 0$ Bosonic modes and all Fermionic modes acquire a “mass” $\sim \pi T$. Clearly, only bosonic $n = 0$ modes are infrared sensitive.



2) Let us consider a vacuum gauge diagram at finite T , where we add a mass term m to keep track of the scale: Linde 80

$$N \text{ loops} \rightarrow \begin{cases} (N-1) & \text{4-vertices} \\ (2N-2) & \text{propagators} \end{cases}$$



$$[T \int d^3p]^N (g^2)^{N-1} \left[\frac{1}{q^2 + m^2} \right]^{2N-2} = \# \times g^6 T^4 \left[\frac{g^2 T}{m} \right]^{N-4}$$

If $m = g^2 T$ (“magnetic” scale), all orders of the loop expansion contribute to pressure at g^6 !

Perturbatively $m = 0$ for magnetic gauge field modes \Rightarrow IR singularity at 4 or larger loop order

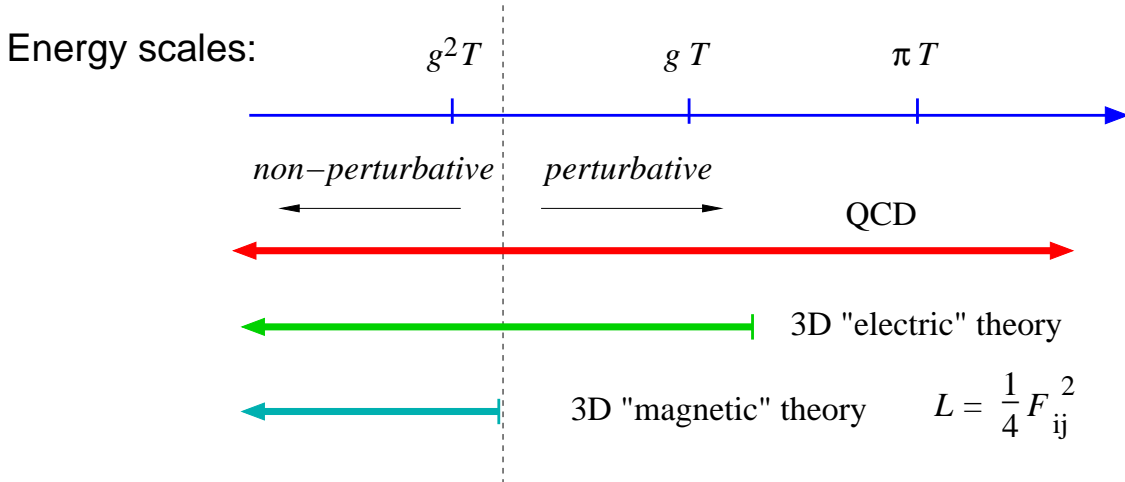
Note: for

$$m \sim \pi T \text{ (hard scales, } n \neq 0)$$

$$m \sim gT \text{ (Debye, electric scales)}$$

the loop expansion is OK.

3d effective theory



We obtain a hierarchy of effective theories

Braaten, Nieto 95

- Integrate over $\pi T \Rightarrow$ 3-dim. effective theory \mathcal{L}_E for gT, g^2T -modes
- Integrate over $gT \Rightarrow$ 3-dim. effective theory \mathcal{L}_M for g^2T -modes

("integrate" = 2-loop optimized matching of theories)

Effective theory \mathcal{L}_E is **3d adjoint Higgs model**:

$$\mathcal{L}_E = \frac{1}{2} \text{Tr} F_{ij}^2 + \text{Tr}[D_i, A_0]^2 + m_D^2 \text{Tr} A_0^2 + \lambda_A (\text{Tr} A_0^2)^2$$

For $N_f = 0$ (pure gauge), the couplings are

$$g_3^2 = g^2 T = \frac{8\pi^2}{11 \ln(6.742T/\Lambda_{\overline{\text{MS}}})} T$$
$$m_D^2 \sim g^2 T^2$$
$$\lambda_A \sim g^4 T$$

\mathcal{L}_E can be analyzed using perturbation theory or lattice simulations. No problems in going to (almost) arbitrarily high T ! (renormalization on the lattice can be done perturbatively)

\mathcal{L}_M is just 3d QCD:

$$\mathcal{L}_M = \frac{1}{2} \text{Tr} F_{ij}^2$$

with coupling constant $g_3^2 \approx g^2 T$.

Pressure:

$$\frac{p}{p_{\text{SB}}} = 1 + g^2 + \underbrace{g^3 + g^4 \ln \frac{1}{g}}_{\mathcal{L}_E} + g^4 + g^5 + g^6 \ln \frac{1}{g} + \underbrace{g^6}_{\mathcal{L}_M} + g^7 + \dots$$

pert. theory ←

The relation between physical pressure and 3d free energy is (Braaten,Nieto)

$$\frac{p}{p_{\text{SB}}} = 1 - \frac{5 \lambda_A}{2 g_3^2} - \frac{45}{8\pi^2} \left(\frac{g_3^2}{T}\right)^3 (\mathcal{F}_E + (\text{known}))$$

where

$$\mathcal{F}_E = -\frac{1}{V g_3^6} \ln \int [dA] \exp[-\int d^3x \mathcal{L}_E]$$

\mathcal{F}_E can now be calculated

- perturbatively $\mapsto g^6 \ln \frac{1}{g}$
- non-perturbatively \mapsto higher order

4-loop graphs for $g^6 \ln \frac{1}{g}$:

of diagrams \sim

3:6:47:490 at 1:2:3:4 loops

One 4-loop diag. with 6 3-point vertices and 9 propagators contains $\sim 24 \times 10^6$ terms, to be integrated over!

$(2^9 \times (3 \times 2)^6)$

Computer algebra program is a *must* to sort out the integrals (FORM)

$$\begin{aligned}
 (\text{skeletons}) = & \frac{1}{72} \text{A} - \frac{1}{4} \text{B} - \frac{1}{6} \text{C} + \frac{1}{12} \text{D} - \frac{1}{2} \text{E} - \frac{1}{2} \text{F} - 1 \text{G} - \frac{1}{3} \text{H} \\
 & + \frac{1}{6} \text{I} + \frac{1}{6} \text{J} + \frac{1}{8} \text{K} - \frac{1}{4} \text{L} + \frac{1}{4} \text{M} - \frac{1}{2} \text{N} + \frac{1}{8} \text{O} + \frac{1}{8} \text{P} \\
 & + \frac{1}{16} \text{Q} + \frac{1}{48} \text{R} + \frac{1}{8} \text{S} + \frac{1}{12} \text{T} - \frac{1}{3} \text{U} + \frac{1}{4} \text{V} + \frac{1}{4} \text{W} + \frac{1}{2} \text{X} \\
 & + \frac{1}{6} \text{Y} + \frac{1}{12} \text{Z} + \frac{1}{2} \text{AA} + \frac{1}{2} \text{AB} + \frac{1}{2} \text{AC} + \frac{1}{8} \text{AD} + \frac{1}{4} \text{AE} + \frac{1}{4} \text{AF} - \frac{1}{2} \text{AG} \\
 & + \frac{1}{4} \text{AH} + \frac{1}{4} \text{AI} + \frac{1}{4} \text{AJ} + 1 \text{AK} + 1 \text{AL} + \frac{1}{4} \text{AM} + \frac{1}{8} \text{AN} + \frac{1}{8} \text{AO} + \frac{1}{2} \text{AP} \\
 & + \frac{1}{2} \text{AQ} + \frac{1}{8} \text{AR} + \frac{1}{4} \text{AS} + \frac{1}{8} \text{AT} + \frac{1}{2} \text{AU} + \frac{1}{2} \text{AV} + \frac{1}{8} \text{AW} + \frac{1}{16} \text{AX} \\
 & + \frac{1}{2} \text{AY} + \frac{1}{16} \text{AZ} + \frac{1}{16} \text{BA} + \frac{1}{6} \text{BB},
 \end{aligned}$$

$$(\text{rings}) = \frac{1}{6} \text{A} + \frac{1}{2} \text{B} + \frac{1}{4} \text{C} - \frac{1}{3} \text{D} - 1 \text{E} - \frac{1}{2} \text{F} + \frac{1}{6} \text{G} + \frac{1}{2} \text{H} + \frac{1}{4} \text{I},$$

$$\text{A} = \frac{1}{2} \text{B} - 1 \text{C} + \frac{1}{2} \text{D} + \frac{1}{2} \text{E} + \frac{1}{2} \text{F},$$

$$\text{B} = 1 \text{C},$$

$$\text{C} = 1 \text{D} + \frac{1}{2} \text{E},$$

$$\begin{aligned}
 \text{D} = & \frac{1}{2} \text{A} - 1 \text{B} - 1 \text{C} - 1 \text{D} + \frac{1}{2} \text{E} + \frac{1}{2} \text{F} + \frac{1}{4} \text{G} \\
 & + \frac{1}{6} \text{H} + \frac{1}{2} \text{I} + \frac{1}{2} \text{J} + \frac{1}{2} \text{K} + 1 \text{L} + 1 \text{M} \\
 & + \frac{1}{2} \text{N} + \frac{1}{2} \text{O} + \frac{1}{4} \text{P} + \frac{1}{4} \text{Q} + \frac{1}{2} \text{R},
 \end{aligned}$$

$$\text{E} = 1 \text{A} + 1 \text{B},$$

$$\text{F} = 1 \text{A} + 1 \text{B} + \frac{1}{2} \text{C} + \frac{1}{2} \text{D} + 1 \text{E} + 1 \text{F} + 1 \text{G} + \frac{1}{2} \text{H},$$

$$\text{G} = 1 \text{A} - 1 \text{B} - 1 \text{C} + \frac{1}{2} \text{D} + 1 \text{E} + \frac{1}{2} \text{F},$$

$$\text{H} = 1 \text{A} + 1 \text{B},$$

$$\text{I} = 1 \text{A} + 1 \text{B} + \frac{1}{2} \text{C}.$$

Non-perturbative contributions

The cleanest way to obtain the genuinely 3-dimensional $g^6 T^4$ contribution is to use \mathcal{L}_M , i.e. 3d QCD. g^6 -term is directly related to the condensate $\langle F_{ij}^2 \rangle$. (Braaten, Nieto 95), (Karsch, Lütgemeier, Patkos, Rank 96)

This can be determined on the lattice. However, this requires 4-loop lattice \leftrightarrow continuum matching:

$$\begin{aligned} \langle F_{ij}^2 \rangle &= \frac{1}{a^3} + \frac{g^2 T}{a^2} + \frac{(g^2 T)^2}{a} \\ &+ (g^2 T)^3 \left(\ln \frac{1}{g^2 T a} + [\text{matching coeff.}] + [\text{non-pert. physics}] \right) \end{aligned}$$

This has not yet been done (stochastic perturbation theory?)

Use \mathcal{L}_E to calculate \mathcal{F}_E :

We cannot calculate the 3-dim. free energy \mathcal{F}_E directly from lattice simulation. However, we can calculate *derivatives* of it:

$$\frac{d\mathcal{F}_E}{dy} = \frac{\partial\mathcal{F}_E}{\partial y} + \frac{\partial x}{\partial y} \frac{\partial\mathcal{F}_E}{\partial x} = \langle A_0^2 \rangle + C \langle A_0^4 \rangle$$

where $y \equiv m_D^2/g_3^4$, $x \equiv \lambda_A/g_3^2$ are dimensionless coupling constants.

Condensates $\langle A_0^2 \rangle$ and $\langle A_0^4 \rangle$ are measurable from the lattice. \mathcal{F}_E is obtained from integral

$$\mathcal{F}_E = \int dy \frac{d\mathcal{F}_E}{dy} + \text{const.}$$

The integration constant is precisely the “Linde term”! We can estimate it by fitting to 4-dim. data.

We drop the condensate $\langle A_0^4 \rangle$ here:

- its contribution is very small (parametrically and numerically)
- we don’t know all lattice counterterms for it

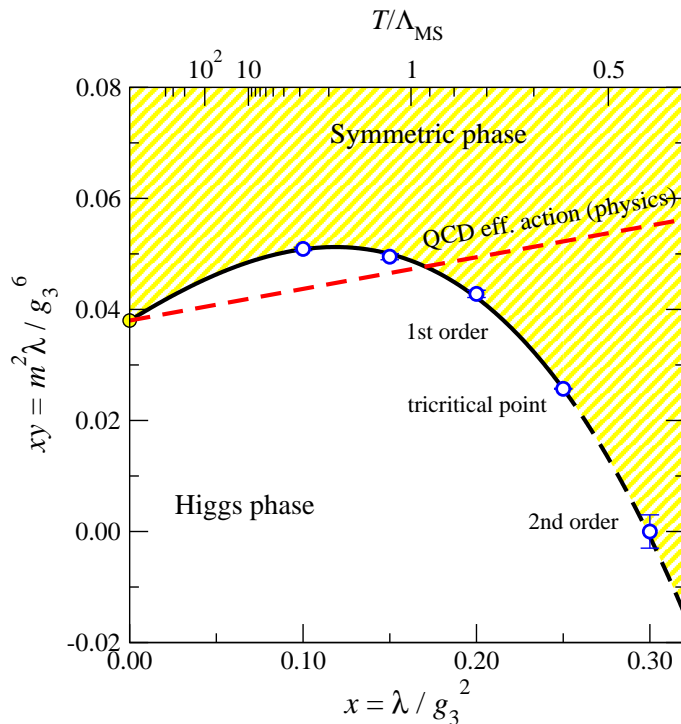
Strategy:

- Measure $\langle A_0^2 \rangle(a)$, subtract *lattice counterterms* (known) and extrapolate to continuum
- Subtract perturbative contributions
- Integrate, result gives $g^7 + g^8 + \dots$ contributions to p
- add perturbative parts of p
- Tune g^6 term, until happy with the fit

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- Bare $\langle A_0^2 \rangle_{\text{latt}}$ has to be determined very precisely: 2 large subtractions! Accuracy to 4–5 decimal digits \rightarrow large volumes (up to 200^3), large statistics.
 - Reliable *continuum extrapolation*: wide range of small lattice spacings $a = 0.37 \dots 0.05 g_3^{-2}$

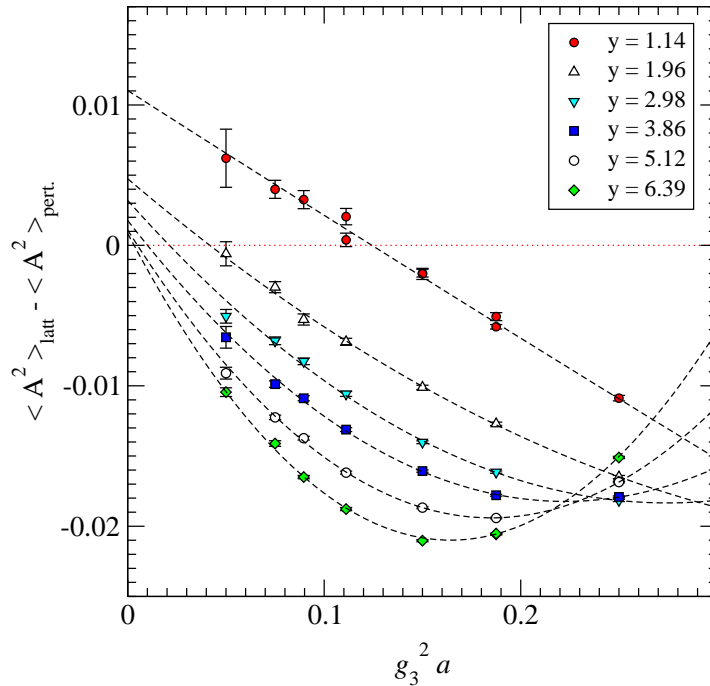
Adjoint Higgs phase diagram

- \exists phase transition, order parameter $\text{Tr } A_0^3$.
- Effective theory must be in the symmetric phase. However, the line which corresponds to QCD lies in the broken phase!
- Solution: the symmetric phase is strongly *metastable*. In practice is not a problem.



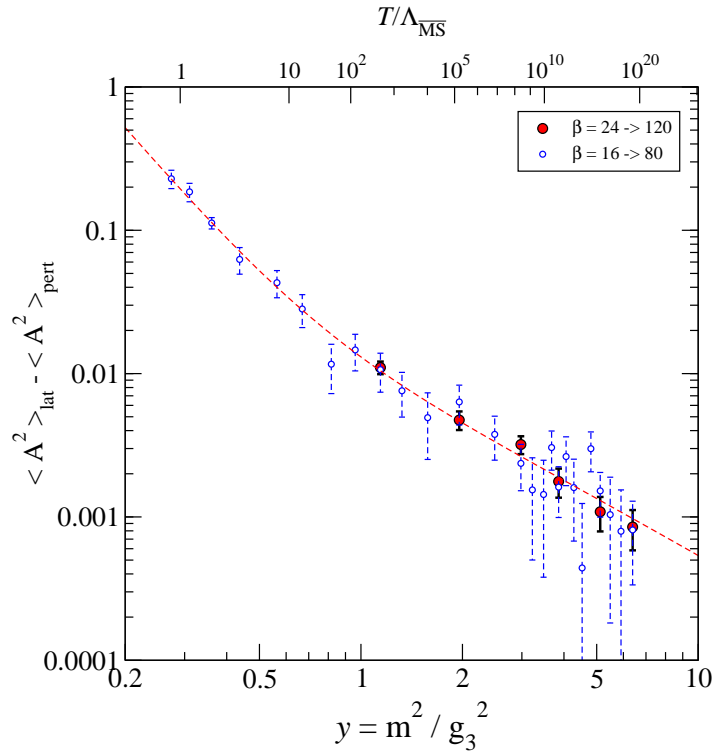
Continuum extrapolation

$y = m_D^2/g_3^4 \sim 1/g^2 = 1.14 - 6.39$, corresponding to $T \sim 100 - 10^{20} \Lambda_{\overline{\text{MS}}}$.



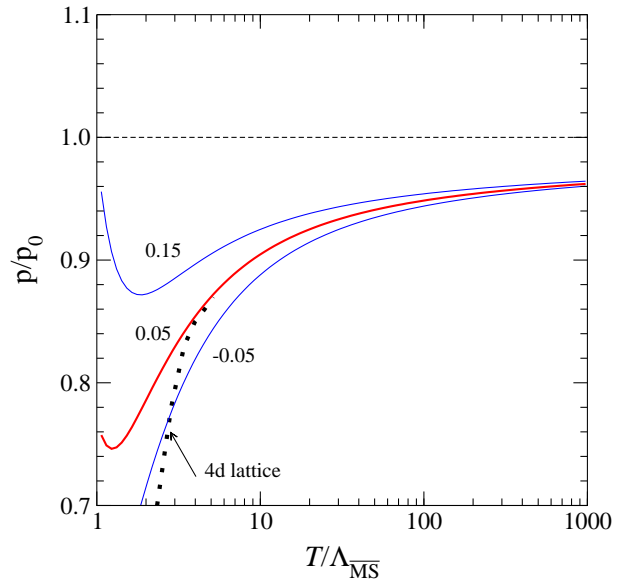
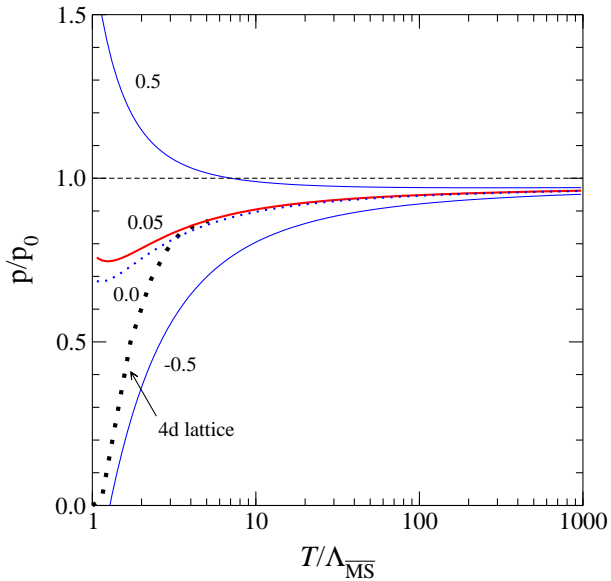
Non-perturbative part of the condensate

Integrate the condensate with an interpolating fit



Pressure

Tune g^6 coefficient to match 4d lattice:



Blue lines on the RHS plot: conservative error estimates for the g^6 coefficient

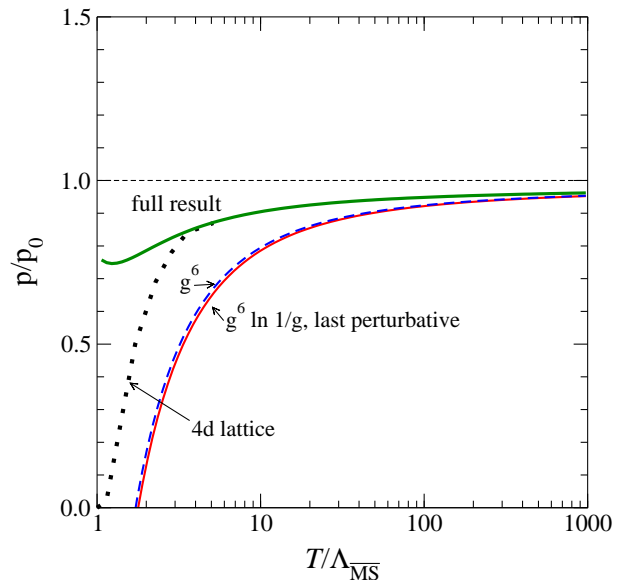
We can now write estimate the full $O(g^6 + g^7 + \dots)$ contribution to p/p_{SB} as

$$0.0373g^6\left(\frac{1}{2}\ln\frac{1}{g} + C\right) - 0.015g^7 + \dots$$

where we estimate $C = -0.05 \dots 0.15$. This is a very small value!

The coefficient of the g^7 -term is not to be taken literally: it matches the data well, but the accuracy is not sufficient to disentangle the higher power coefficients separately.

Pressure, with full perturbative and non-perturbative contributions:



Conclusions

- Full expression for pressure in pure gauge QCD:

$$p(T) = p_{\text{pert.}}(T) + p_{\text{non-pert}}(T)$$

- When perturbative expansion is organized as here, the coefficient of $g^6 T^4$ -term is very small.
- Smallness of the $g^6 T^4$ coefficient here does not imply that the free energy of \mathcal{L}_M , 3d QCD, is small! The coefficient is a sum of purely magnetic modes (\mathcal{F}_M) and other contributions, some perturbative, some not.
- The method used here relies on the accuracy of a) 4d lattice results, and b) \mathcal{L}_E down to $T \sim 4T_c$.
However, even allowing for quite conservative errors, the conclusions do not change qualitatively.
- Direct determination of the $g^6 T^4$ contribution \mathcal{L}_M is not restricted by these limitations.