Notes on algebras, representations and Young calculus

Let G be a Lie group (i.e., a continuous group whose elements can be parametrized in terms of real parameters, for which the group manifold is differentiable). The *generators* of G are denoted by t_a, and can be defined by looking at elements $q \in G$, which are infinitesimally close to the identity element:

$$q = \mathbb{1} + i \sum_a x_a t_a + \mathcal{O}(x^2)$$

where the summation index ranges from 1 to the dimension of the group, and the x_a's are infinitesimally small parameters.

The generators t_a form a basis for a vector space (called the "algebra of generators of G", and usually denoted as g), in which it is possible to define an internal "product" operation denoted as $[\ ,\ ]$ ("Lie bracket"), which is linear in each of its two arguments, and satisfies the following properties:

1. the Lie bracket is "alternating in g": $\forall u, v \in g: \quad [u, u] = 0$. Due to the bilinearity, this condition is equivalent to antisymmetry: $\forall u, v \in g: \quad [u, v] = -[v, u]$
2. "Jacobi identity": $\forall u, v, w \in g: \quad [[u, v], w] + [[w, u], v] + [[v, w], u] = 0$

In a representation of the group G in terms of square matrices of size $N \times N$ (in which, obviously, also the elements of g are square matrices of the same size), the Lie bracket can be expressed as the *commutator* of two matrices:

$$[A, B] = AB - BA \tag{1}$$

Since the Lie product is closed in the algebra, and since the t_a's are a basis for the algebra itself, Lie products of different generators can be expressed as linear combinations of the t_a's themselves, namely:

$$[t_a, t_b] = i \sum_c C_{abc} t_c \tag{2}$$

where the C_{abc} are called the "structure constants" of the algebra; owing to the properties of the Lie bracket, they satisfy certain conditions (for example, the antisymmetry of the Lie bracket implies that $C_{abc} = -C_{bac}$).

As an example, if we consider the SU(2) group in its fundamental (defining) representation in terms of 2×2 unitary complex matrices with unit determinant, it turns out that the generators can be written in terms of the three Pauli matrices σ_a ($a \in \{1, 2, 3\}$) as:

$$t_a = \frac{1}{2}\sigma_a$$

with:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

With this definition of the t_a's, it is possible to show that:

$$[t_a, t_b] = i \sum_{c=1}^{3} \epsilon_{abc} t_c \qquad (3)$$

(where $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, while $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$, and $\epsilon_{abc} = 0$ if at least two indices are equal) where the structure constants of $su(2)$ appear: $C_{abc} = \epsilon_{abc}$.

Note that, with the conventions introduced above, the t_a's of a generic $su(N)$ algebra can be represented as Hermitean traceless matrices. The fact that the trace of the t_a's vanishes is related to the requirement that we are dealing with a *special* unitary group, i.e. that the determinant of the group elements is 1.

The structure constants are at the very core of the geometric structure of an algebra: essentially, defining an algebra means defining its structure constants. In particular, this implies that an algebra can be realized using different matrices of different sizes: the relations defined by eq. (2) could be, for example, satisfied by a set of matrices of size $N_1 \times N_1$, or by a different set of matrices of size $N_2 \times N_2$. This corresponds to different "representations" of the algebra.

By constructing different representations of an algebra, it is also possible to obtain the corresponding representations of the group, of the same size: for group elements infinitesimally close to the identity element, the group and algebra elements are related by eq. (1), which, for a generic element q of the group, generalizes to the *exponential map*:

$$q = \exp\left(i \sum_a x_a t_a \right) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(i \sum_a x_a t_a \right)^n$$

If the group elements are represented as (real or complex) matrices of size $N \times N$, they can be considered as linear transformations acting on a N-component (real or complex) vector space.

Many problems in physics have to do with the properties of a *system*, which is made of several, simpler, objects. If each component of the system has well-defined transformation properties with respect to a certain group of transformations (meaning, for example, that it transforms according to a particular representation of the group), it is often useful to work out which are the transformation properties of the system as a whole. One example could be the computation of the total spin of a quantum system of two non-relativistic, indistinguishable particles: if one of the constituents transforms as an N-component real or complex vector under the given group of transformations, and the other transforms as an M-component vector, then the compound system of the two particles could be represented as a vector with $N \cdot M$ components, given by the products of the components of the two original vectors. The action of a transformation in the group on such a vector can then be described via the "tensor product" of the two matrices that describe the transformations of the two original vectors, which is a $(N \cdot M) \times (N \cdot M)$ matrix with the structure:

$$
\begin{aligned}
A \otimes B &= \begin{pmatrix} a_{11} & \dots & a_{1N} \\ \dots & \dots & \dots \\ a_{N1} & \dots & a_{NN} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & \dots & b_{1M} \\ \dots & \dots & \dots \\ b_{M1} & \dots & b_{MM} \end{pmatrix} = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1N}B \\ \dots & \dots & \dots & \dots \\ a_{N1}B & a_{N2}B & \dots & a_{NN}B \end{pmatrix} \\
&= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & \dots & a_{11}b_{1M} & a_{12}b_{11} & \dots & a_{1N}b_{1M} \\ a_{11}b_{21} & a_{11}b_{22} & \dots & a_{11}b_{2M} & a_{12}b_{21} & \dots & a_{1N}b_{2M} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{N1}b_{M1} & a_{N1}b_{M2} & \dots & a_{N1}b_{MM} & a_{N2}b_{M1} & \dots & a_{NN}b_{MM} \end{pmatrix}
\end{aligned}
$$

(Note that in general the tensor product of two matrices is not commutative: $A \otimes B \neq B \otimes A$, but the two resulting matrices can be mapped into each other by a permutation of rows and columns.)

If A is a matrix of size $N \times N$, and B is of size $M \times M$, it is easy to prove that: $\mathrm{tr}(A \otimes B) = (\mathrm{tr}A) \cdot (\mathrm{tr}B)$, and $\det(A \otimes B) = (\det A)^M \cdot (\det B)^N$.

A different operation on matrices is the "tensor sum", which can be used to describe the case of two transformations acting separately on two independent vectors: in this case, the result is a matrix of size $(N + M) \times (N + M)$:

$$
A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1N} & 0 & \dots & 0 \\ a_{21} & \dots & a_{2N} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{N1} & \dots & a_{NN} & 0 & \dots & 0 \\ 0 & \dots & 0 & b_{11} & \dots & b_{1M} \\ 0 & \dots & 0 & b_{21} & \dots & b_{2M} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & b_{M1} & \dots & b_{MM} \end{pmatrix}
$$

(Similarly to what happens for the tensor product of two matrices, also the "tensor sum" of two matrices is not commutative: $A \oplus B \neq B \oplus A$, but, again, the two resulting matrices can be mapped into each other by a permutation of rows and columns.)

It is obvious that: $\mathrm{tr}(A \oplus B) = \mathrm{tr}A + \mathrm{tr}B$, and $\det(A \oplus B) = (\det A) \cdot (\det B)$.

The operations above are useful in the composition of different representations. In particular, it is interesting to look at the tensor products of different *irreducible* representations (which, roughly speaking, are those that cannot be written in terms of matrices in a block-triangular or block-diagonal form), and to decompose them in the sum of other irreducible representations. To do this, it is useful to use the rules of Young calculus, which we describe below. For simplicity, we only discuss the cases relevant for the algebras of generators of unitary or special unitary groups, $u(N)$ and $su(N)$ respectively.

- Each representation is denoted by a Young diagram made of square boxes, arranged in horizontal rows and vertical columns.

- The lengths of the horizontal rows of a Young diagram are non-increasing, from top to bottom.

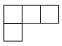
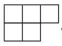
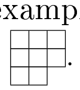
- The maximum number of horizontal rows in the Young diagram of a representation of $u(N)$ is N; for $su(N)$, the length of the N-th row is always zero.

The box diagrams can also be represented by a sequence of N non-increasing integers $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{N-1} \geq \lambda_N$, with $\lambda_N = 0$ for $su(N)$, that represent the lengths of subsequent rows: $[\lambda_1, \ldots, \lambda_N]$ (sometimes the shortened notation $[\lambda_1, \ldots, \lambda_{N-1}]$ is used for $su(N)$).

Young diagrams of particularly interesting representations include:

- The Young diagram of the fundamental representation consists of only one box.

- The Young diagram of the trivial representation (in which each generator t_a is represented by the number 0, and each element of the group is represented by 1) does not contain any box: it can denoted by the symbol $\emptyset$.

- Given an irreducible representation r whose first row has length λ_1, the Young diagram of the "conjugate representation" $\bar{r}$ is obtained from the rectangle of N rows and λ_1 columns in which the Young diagram of r can be inscribed, by removing the boxes belonging to the Young diagram of r, and by rotating the diagram made of the remaining boxes by 180 degrees. For example, for $su(3)$, the conjugate representation of  is , whereas for $su(4)$ it is .

- In particular, the Young diagram of the "anti-fundamental" representation is given by a vertical column of $N-1$ boxes.

- If the Young diagram of the conjugate representation is equal to the one of the initial representation, then the latter is self-conjugate; in particular, this implies that the trace of all group elements in that representation are real.

- The Young diagram of the "adjoint representation" is made of a vertical column of $N-1$ boxes, together with a further box in the first horizontal row; the adjoint representation is always self-conjugate.

Given a non-empty Young diagram, the size of the corresponding representation (i.e., the dimension of the vector space on which the representation acts) is given by the number of possible ways to fit the numbers from 1 to N in each of the boxes of the diagram, with the constraints that:

- In every horizontal row, the sequence of numbers from left to right must be non-decreasing.

- In every vertical column, the sequence of numbers from top to bottom must be strictly increasing.

Roughly speaking, higher representations are obtained from the indices $1, \ldots, N$ of the fundamental one, either by symmetrization (of indices in the same horizontal row) or by antisymmetrization (of indices in the same vertical column). The two rules listed above enforce these properties and help avoiding multiple-countings.

Equivalently, the dimension of the representation $[\lambda_1, \lambda_2, \ldots, \lambda_N]$ is given by:

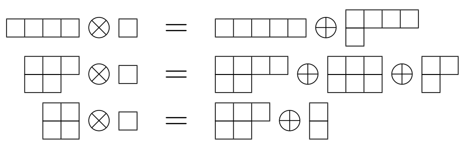
$$\prod_{i=1}^{N-1} \prod_{j=i+1}^{N} \frac{l_i - l_j}{l_i^0 - l_j^0}, \quad \text{with: } l_i = \lambda_i + N - i, \quad l_i^0 = N - i$$

(with $\lambda_N = 0$ for $su(N)$).

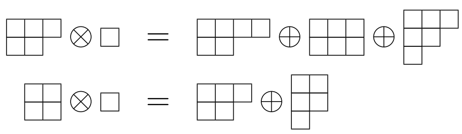
Using the Young diagrams, it is possible to decompose generic tensor products of different representations into a direct sum of irreducible representations: this is called the Young calculus. Using the properties of the traces of $A \otimes B$ and $A \oplus B$, the Young calculus allows one to compute the characters of group elements in different representations.

The tensor products of representations that are easiest to simplify are those for which one of the two factors is the fundamental representation (for convenience, it is easiest to have the fundamental representation as the second factor—this does not change the representation decomposition of the product). They are decomposed into a direct sum of new representations, which are obtained by adding the box of the fundamental representation in each of the rows of the initial diagram, with the constraint that resulting diagrams should still have rows on non-increasing length. For $su(N)$, if in this process the first column gets completely filled, it can be deleted from the corresponding diagram.

Some examples, for $su(3)$:



Note that for $u(N \geq 3)$, or for $su(N \geq 4)$, the latter two products yield instead:



When neither factor is the fundamental representation, the Young calculus rules get considerably more cumbersome; again, it is easiest to have the representation with the simplest (smallest) Young diagram as the second factor. Then the construction of the representations appearing in the decomposition of the product goes as follows:

1. Replace the boxes appearing in the Young diagram of the second factor by letters, with the following rules: a's in the first row, b's in the second row, et c.

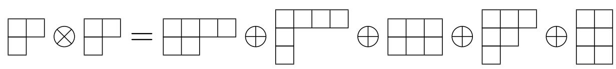
2. Attach the a's to the right end of the rows of the Young diagram appearing as the first factor in all possible ways, but without placing any two a's in the same column.

3. Repeat the previous step for the b's, then the c's, et c.

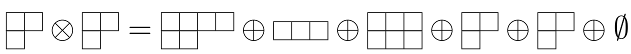
4. At the end, for each of the possible diagrams obtained, read the sequence of letters *from right to left* and from top to bottom, and discard those representations which violate the following *admissibility rule*: at any point in the sequence, at least as many a's have occurred as b's, at least as many b's have occurred as c's, et c. (for example: the sequences aab and aba are admissible, while baa is not).

5. In the diagrams that satisfy the admissibility rule, replace the a's, b's c's, et c. with boxes to get the Young diagrams of the decomposition into a direct sum of representations (for $su(N)$, cancel out possible columns of N boxes, if any).

An example (for $su(N > 3)$, or for $u(N > 2)$) is the following:



Note that, according to the rules above, for $su(3)$ the same product would yield:



Finally, as a cross-check of the representation composition rules, it may be useful to evaluate the size of the different representations: interpreting the representation composition operations in terms of the matrix product and sums discussed above, the matrix sizes on the l.h.s. and r.h.s. of any decomposition should match. For example, in eq. (4) for $su(3)$ we have:

$$8 \cdot 8 = 27 + 10 + 10 + 8 + 8 + 1$$

Note, however, that this is a necessary but not sufficient condition for the correctness of the decomposition.