Classification of simple Lie algebras

Simple roots of a simple Lie algebra are such that:

1. they are linearly independent vectors;
2. if α and β are simple roots, then:

$$-\frac{2\alpha \cdot \beta}{\alpha^2} \in \mathbb{N}; \tag{1}$$

3. the set of simple roots cannot be split into non-trivial disjoint subsets, such that every element from one of them is orthogonal to every element from another subset.

These conditions define a so-called "Π-system". Π-systems can be associated with Dynkin diagrams, made of circles (representing simple roots), in which all possible pairs of circles are either connected by no line (if the angle between the associated simple roots is $\pi/2$), one line (if the angle between the corresponding simple roots is $2\pi/3$), two lines (if the angle is $3\pi/4$) or three lines (if the angle is $5\pi/6$). Furthermore, in the case of diagrams with a double or a triple line, the simple roots have different lengths (in particular, the ratio between the lengths is $\sqrt{2}$): in that case, short simple roots are denoted by filled circles, and/or are pointed at by an arrow on the double or triple line.

The linear independence of simple roots (together with the fact that the only allowed angles between simple roots are $\pi/2$, $2\pi/3$, $3\pi/4$ and $5\pi/6$, and with the fact that the sum of the angles between three linearly independent vectors in $\mathbb{R}^3$ is always less than 2π) and the requirement that a Π-system must be connected (hence no simple root can be at an angle $\pi/2$ with all of the others) imply that the only Π-systems of three simple roots are: Because of this, any subset of a Dynkin diagram must still be a Dynkin diagram, thus

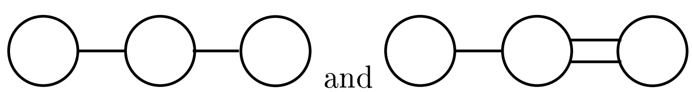


Figure 1: The only Π-systems made of three simple roots.

Dynkin diagrams with three or more roots can only consist of pieces of the two forms in the figure above.

In particular, this implies that triple lines can only appear in a diagram of two simple roots: This diagram corresponds to the exceptional algebra G_2, which has rank 2 and dimension 14; one of its two simple roots is shorter than the other, and this is denoted by the filled circle (and pointed at by the arrow).

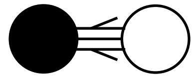
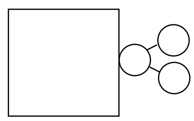


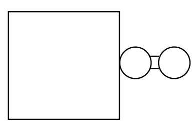
Figure 2: The only Π-system including a triple line.

Furthermore, it is possible to prove that, in any Dynkin diagram, the operation of merging two roots which are connected by a single line leads to another Dynkin diagram. Thus, a Π-system cannot have two or more than two double lines (otherwise, by merging all roots connected by single lines, it could be reduced to a three-root diagram with two double lines, which is not allowed), and it cannot have any loop (otherwise, by merging enough roots connected by single lines, it could be reduced to a loop three-root diagram, which is not allowed).

It is also possible to prove that, if a Π-system is of the form: (where the rectangle



generically denotes the rest of the diagram) then also: is a Π-system. This implies that, if



in a Π-system there are branches, they can only have the form of three single lines coming out of the simple root in the center: Otherwise, diagrams with four or more branches, and/or with branches and double lines, and/or with more branchings stemming out of different simple roots, could be shrunk to a three-root diagram with two double lines, which is not allowed.

Finally, no Π-system can be (or include as a subdiagram) any of the following diagrams: This is so, because it can be proven that for all of them there exists a vanishing linear combination of the simple roots with non-vanishing integer coefficients (i.e., their simple roots are not linearly independent). Note that, in the diagram with the double line, for which one can choose the direction in which the arrow points (i.e. which simple roots are short and which are long), both possible choices of the long and short simple roots can be proven to violate the requirement of linear independence.

This leads to the following classification of all possible Dynkin diagrams: there are four infinite series A_n, B_n, C_n and D_n (where n denotes the rank, i.e. the number of simple roots—which is also equal to the number of Cartan generators), which are associated to the algebras of classical simple Lie groups, and five exceptional diagrams: G_2, F_4, E_6, E_7 and E_8, associated to corresponding exceptional Lie groups. In particular:

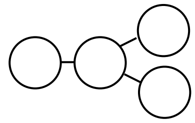


Figure 3: Π-systems can have at most one branching, in the form of three single lines coming out of a simple root.

Figure 4: Diagrams which are forbidden, because they violate the condition of linear independence of the simple roots.

- A_n diagrams are chains of n simple roots, connected by $(n-1)$ single lines and

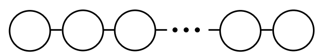


Figure 5: Structure of the Dynkin diagrams for the A_n algebras.

correspond to the algebras of the $SU(n+1)$ groups, with rank n and dimension $n(n+2)$;

- B_n diagrams are chains of n simple roots, connected by $(n-2)$ single lines, and by one double line at one end of the chain; all simple roots are long, except the one that is only connected to the others by the double line. These diagrams correspond to the algebras of the $SO(2n+1)$ groups, with rank n and dimension $n(2n+1)$;

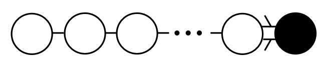


Figure 6: Structure of the Dynkin diagrams for the B_n algebras.

- C_n diagrams have the same structure as B_n diagrams, except that all simple roots are short, except the one that is only connected to the others by the double line. These diagrams corresponds to the algebras of the $Sp(2n)$ groups, with rank n and

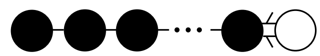


Figure 7: Structure of the Dynkin diagrams for the C_n algebras.

dimension $n(2n+1)$;

- D_n diagrams are chains of n simple roots, in which the first $(n-2)$ simple roots are connected by single lines, and the simple root at one of the ends of the chain is also attached to two further simple roots, by single lines; These diagrams correspond to

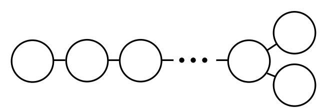


Figure 8: Structure of the Dynkin diagrams for the D_n algebras.

the algebras of the $SO(2n)$ groups, with rank n and dimension $n(2n-1)$;

- G_2 is a diagram with two simple roots connected by a triple line. One of the two simple roots is short. The rank of G_2 is 2 and its dimension is 14;

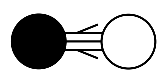


Figure 9: Dynkin diagram of the G_2 algebra.

- F_4 is a diagram which is a chain of four simple roots: they are connected by two single lines and by one double line (at the center of the chain). The first two (or, equivalently, the third and the fourth) simple roots are short. The rank of the F_4 algebra is 4 and its dimension is 52;
- E_6 is a diagram consisting of 6 simple roots: 5 of them form a linear chain, connected by single lines, and the last one is connected (again, by a single line) to the third root in the chain. The rank of E_6 is 6 and its dimension is 78;
- E_7 is a diagram similar to E_6, except that it includes 7 simple roots: 6 of them form a linear chain, connected by single lines, and the last one is connected by a single line to the third root in the chain. The rank of E_7 is 7, and its dimension is 133;

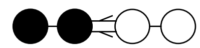


Figure 10: Dynkin diagram of the F_4 algebra.

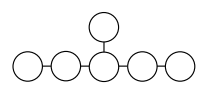


Figure 11: Dynkin diagram of the E_6 algebra.

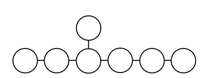


Figure 12: Dynkin diagram of the E_7 algebra.

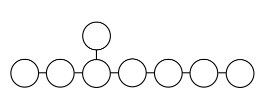


Figure 13: Dynkin diagram of the E_8 algebra.

- E_8 is a diagram made of 8, in which 7 of them form a linear chain, connected by single lines, and the last one is connected by a single line to the third root in the chain. The rank of E_8 is 8, and its dimension is 248.

Note that, for the algebras represented by the diagrams of type A, D and E, all simple roots have the same length: these algebras are said to be "simply laced".