Notes on homotopy groups and exact sequences

Given a topological space X, its different homotopy groups $\pi_q(X)$, for $q \in \mathbb{N}$, classify the homotopically inequivalent ways to map a S^q sphere (which can be obtained from the q-fold product of intervals $[0,1] \subset \mathbb{R}$:

$$I^q = \underbrace{[0,1] \times [0,1] \times \dots [0,1]}_{q \text{ factors}} \subset \mathbb{R}^q$$

by identifying all the points on its boundary) to X, where "homotopically inequivalent" means "that cannot be mapped into each other by continuous deformations". In particular:

- $\pi_0(X)$ is the set of connected components of X,
- $\pi_1(X)$ is the set of homotopically inequivalent loops in X,
- $\pi_2(X)$ is the set of homotopically inequivalent closed surfaces in X,

et c. (Note, in particular, that, since a generic $\mathbb{Z}_N$ consists only of N points, it does not contain any one-, two- or three-loop or any higher loops either, and the only possibly non-trivial homotopy group is $\pi_0(\mathbb{Z}_N) \cong \mathbb{Z}_N$.)

The homotopy groups of direct products of (group) manifolds are given by the direct sum of the corresponding homotopy groups of each factor:

$$\pi_q(X_1 \times X_2) = \pi_q(X_1) \oplus \pi_q(X_2)$$

(where the meaning of the direct sum on the right-hand side of this equation is that the two homotopy groups are independent of each other).

Of particular interest are the homotopy groups of N-dimensional spheres. Recall that: $S^N = \mathrm{O}(N+1)/\mathrm{O}(N) = \mathrm{SO}(N+1)/\mathrm{SO}(N)$ (and, furthermore, for spheres of odd dimension larger than or equal to three, one also has: $S^{2k+1} = \mathrm{U}(k+1)/U(k)$); it turns out that:

$$\pi_q(S^N) = \begin{cases} \mathbb{Z}_1 & \text{for } q < N \\ \mathbb{Z} & \text{for } q = N \end{cases}$$

(where $\mathbb{Z}_1 = \{e\}$ denotes the trivial group, containing only the identity element). For $q > N$, in general $\pi_q(S^N)$ can be non-trivial; in particular, the simplest non-trivial case is $\pi_3(S^2) = \mathbb{Z}$ (which is related to the "Hopf fibration").

In order to compute the homotopy groups of various manifolds, it is often useful to resort to exact sequences of group homomorphisms.

A generic sequence of groups G_i and group homomorphisms $f_i : G_i \to G_{i+1}$:

$$\ldots \longrightarrow G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} G_{i+2} \longrightarrow \ldots$$

is said to be "exact" if, for every a (except, possibly, at the end of the sequence), one has: $\mathrm{Im}\ f_a \cong \mathrm{Ker}\ f_{a+1}$, where the symbol $\cong$ denotes group isomorphism.

Furthermore, recall that, due to a fundamental theorem of group homomorphisms, given a group homomorphism f defined on a group G, one has:

$$\mathrm{Im}\ f = G/\mathrm{Ker}\ f.$$

Given a Lie group G, and a compact Lie subgroup $H \subset G$, it is possible to prove that the following sequence:

$$\cdots \to \pi_q(H) \to \pi_q(G) \to \pi_q(G/H) \to \pi_{q-1}(H) \to \pi_{q-1}(G) \to \pi_{q-1}(G/H) \to \ldots$$

(constructed by repeating the basic block: $\cdots \to \pi_m(H) \to \pi_m(G) \to \pi_m(G/H) \to \ldots$ for values of m which decrease by 1 every time) is exact.

Typically, the determination of non-trivial homotopy groups using exact sequences can be done by:

- considering a portion of the sequence starting and ending from the trivial group $\mathbb{Z}_1$,
- using the definition of exact sequence,
- using the group homomorphisms' theorem.

As an example, consider the computation of $\pi_3\left(\mathrm{SO}(3)\right)$; since $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$, we can write:

$$\pi_3(\mathbb{Z}_2) \xrightarrow{f} \pi_3\left(\mathrm{SU}(2)\right) \xrightarrow{g} \pi_3\left(\mathrm{SO}(3)\right) \xrightarrow{h} \pi_2(\mathbb{Z}_2)$$

First of all, we have: $\pi_3(\mathbb{Z}_2) \cong \pi_2(\mathbb{Z}_2) \cong \mathbb{Z}_1$. Second, note that $\pi_3\left(\mathrm{SU}(2)\right) \cong \mathbb{Z}$, because $\mathrm{SU}(2)$ is isomorphic to S^3. Next, one can observe that h necessarily maps every element of its domain $\pi_3\left(\mathrm{SO}(3)\right)$ to the unique element of $\pi_2(\mathbb{Z}_2) \cong \mathbb{Z}_1$, which is the identity element of $\mathbb{Z}_1$: hence, $\mathrm{Ker}\ h \cong \pi_3\left(\mathrm{SO}(3)\right)$, and, since the sequence is exact, one obtains: $\mathrm{Im}\ g \cong \mathrm{Ker}\ h \cong \pi_3\left(\mathrm{SO}(3)\right)$. Due to the homomorphism theorem, one also has: $\mathrm{Im}\ g \cong \pi_3\left(\mathrm{SU}(2)\right)/\mathrm{Ker}\ g$. Given that the sequence is exact, one has: $\mathrm{Ker}\ g \cong \mathrm{Im}\ f$, but, since the domain of f is isomorphic to $\mathbb{Z}_1$, and f is a homomorphism, f necessarily maps the unique element of its domain to the identity element in $\pi_3\left(\mathrm{SU}(2)\right)$, so $\mathrm{Im}\ f \cong \mathbb{Z}_1$. Thus: $\mathrm{Ker}\ g = \mathbb{Z}_1$. Therefore we find that:

$$\begin{aligned}\pi_3\left(\mathrm{SO}(3)\right) &\cong \mathrm{Ker}\ h \cong \mathrm{Im}\ g \cong \pi_3\left(\mathrm{SU}(2)\right)/\mathrm{Ker}\ g \cong \pi_3\left(\mathrm{SU}(2)\right)/\mathrm{Im}\ f \\ &\cong \pi_3\left(S^3\right)/\mathrm{Im}\ f \cong \pi_3\left(S^3\right)/\mathbb{Z}_1 \cong \mathbb{Z}/\mathbb{Z}_1 \cong \mathbb{Z}.\end{aligned}$$