

Some miscellaneous definitions and formulæ

★ A magma is defined as a set M equipped with a binary operation $M \times M \rightarrow M$.

★ A (partial) order relation \preceq in a set X is a binary relation among elements of X which, $\forall a, b, c \in X$, satisfies:

- $a \preceq a$
- $[(a \preceq b) \text{ AND } (b \preceq a)] \Rightarrow (a = b)$
- $[(a \preceq b) \text{ AND } (b \preceq c)] \Rightarrow (a \preceq c)$

namely, which is reflexive, antisymmetric, and transitive.

★ Given a Lie algebra \mathcal{A} and a subalgebra $\mathcal{I} \subset \mathcal{A}$, we say that \mathcal{I} is an ideal, when, for all $x \in \mathcal{A}$ and for all $y \in \mathcal{I}$, one has: $[x, y] \in \mathcal{I}$.

★ A Lie algebra is said to be semisimple, if it does not contain non-trivial Abelian ideals.

★ A Lie group G is said to be semisimple if its Lie algebra is semisimple, or, equivalently, if G does not have any non-trivial connected, normal, Abelian subgroups.

★ The zero-th homotopy group of a topological space M , denoted by $\pi_0(M)$ is the number of disconnected components of M .

★ The n -dimensional unit pseudospheres are n -dimensional hypersurfaces of constant curvature in the $(n + 1)$ -dimensional real vector space \mathbb{R}^{n+1} . There are four of them:

1. the n -dimensional sphere S^n , defined as the set of points $x \in \mathbb{R}^{n+1}$ satisfying the equation:

$$x_0^2 + (x_1^2 + x_2^2 + \cdots + x_{n-1}^2) + x_n^2 = 1,$$

2. the n -dimensional hyperbolic space H^n , described by the equation:

$$-x_0^2 + (x_1^2 + x_2^2 + \cdots + x_{n-1}^2) + x_n^2 = -1,$$

3. the n -dimensional de Sitter space dS_n , described by the equation:

$$-x_0^2 + (x_1^2 + x_2^2 + \cdots + x_{n-1}^2) + x_n^2 = 1,$$

4. the n -dimensional anti-de Sitter space AdS_n , described by the equation:

$$-x_0^2 + (x_1^2 + x_2^2 + \cdots + x_{n-1}^2) - x_n^2 = -1,$$

where x_0, \dots, x_n denote the components of a generic element x of \mathbb{R}^{n+1} .

★ Let A be a generic, diagonalizable matrix. Then:

$$\det e^A = e^{\operatorname{tr} A}.$$

To prove this statement, recall that:

$$e^A = \sum_{n=0}^{+\infty} \frac{A^n}{n!}.$$

If A is diagonal ($A = \operatorname{diag}(\lambda_1, \lambda_2, \dots)$), then it is trivial to check that: $A^n = \operatorname{diag}(\lambda_1^n, \lambda_2^n, \dots)$, and, hence: $\exp A = \operatorname{diag}(e^{\lambda_1}, e^{\lambda_2}, \dots)$, which implies:

$$\det e^A = \prod_i (\exp A)_{ii} = \prod_i e^{\lambda_i} = \exp\left(\sum_i \lambda_i\right) = e^{\operatorname{tr} A}.$$

On the other hand, if A is not diagonal, then, since it is diagonalizable, there exists a similarity transformation that diagonalizes it:

$$A = U^{-1}\Lambda U,$$

where $\det U \neq 0$, and Λ is diagonal. Then: $\det A = \det(U^{-1}\Lambda U) = \det(U^{-1}) \cdot \det \Lambda \cdot \det U = (\det U)^{-1} \cdot \det \Lambda \cdot \det U = \det \Lambda$, and the previous argument can be applied to Λ . Note that, if $A = U^{-1}\Lambda U$, then $\operatorname{tr} A = \operatorname{tr}(U^{-1}\Lambda U) = \operatorname{tr}(UU^{-1}\Lambda) = \operatorname{tr} \Lambda$, due to the invariance of the trace of a product of matrices, under cyclic permutations of the factors.