Problem set 12

Answer the following questions, giving motivations for each answer, and return the solutions to Tuukka Meriniemi's mailbox (corridor A on the third floor of the Physicum building).

1. In the five-dimensional real vector space $\mathbb{R}^5$, with Cartesian coordinates (x, y, z, t, w), let S be the set of points whose coordinates simultaneously satisfy all of the following conditions:
$$\begin{cases} (\sinh w)(\cosh t) < (\sinh t)(\cosh w) \\ \left(t - \sqrt{x^2+y^2}\right)^2 = -z^2 + w^2 \\ (tw)^3 \geq 0 \\ 1 > \int_{-\infty}^{+\infty} \frac{e^{-k^2}}{\Gamma(1/2)e^{t^2k^2}} dk \\ t \geq 0 \end{cases} .$$
Furthermore, let X_0 be the subset of $\mathbb{R}^2$ defined by the condition that, for each of its elements $(t_0, w_0) \in X_0$, there exists at least one element of S, having t_0 as its fourth coordinate, and w_0 as its fifth coordinate. For a given $(t_0, w_0) \in X_0$, let $P_{(t_0,w_0)}$ be the subset of $\mathbb{R}^3$ defined by the condition that a generic $(x, y, z) \in \mathbb{R}^3$ is an element of $P_{(t_0,w_0)}$ if and only if (x, y, z, t_0, w_0) is an element of S.
   - What is the dimension of $P_{(t_0,w_0)}$? Does the answer depend on (t_0, w_0) or not?
   - Can you find any particularly simple parametrization(s) for the points of $P_{(t_0,w_0)}$, in terms of angular coordinates ?
   - If there exists a case, for which the dimension of $P_{(t_0,w_0)}$ is two, and for which you worked out a parametrization of $P_{(t_0,w_0)}$ in terms of two angular coordinates θ and ϕ, compute the corresponding components of the induced metric: $g_{\theta\theta}(\theta, \phi)$, $g_{\theta\phi}(\theta, \phi)$, and $g_{\theta\theta}(\theta, \phi)$, assuming that $\mathbb{R}^3$ is endowed with the Euclidean metric.
   - If there exists a case, for which the dimension of $P_{(t_0,w_0)}$ is three, and for which you worked out a parametrization for its points in terms of three angular coordinates, compute the corresponding components of the induced metric, assuming that $\mathbb{R}^3$ is endowed with the Euclidean metric.
   - Compute the first homotopy group of $P_{(t_0,w_0)}$, and discuss whether it depends on (t_0, w_0) or not.

2. Let G be a Lie group and g the algebra of its generators; compute the number of roots and the number of simple roots of g for the following cases:

- $G = \mathrm{SU}(14)$
- G is the group of transformations that map different orthonormalized, centered-at-the-origin reference frames in the twelve-dimensional real vector space into each other
- G is the group of orientation-preserving transformations that map different orthonormalized, centered-at-the-origin reference frames in the twelve-dimensional real vector space into each other
- G is the group of 8×8 matrices A with real entries, which satisfy the property:

$$BA^T B = -A^{-1}$$

  where: $B = i\sigma_2 \otimes \left((\sigma_1)^2\right) \otimes \left((\sigma_3)^2\right)$
- G is the largest simple group among those whose algebra is characterized by the fact that three distinct pairs of simple roots are mutually orthogonal, two distinct pairs of simple roots form a $2\pi/3$ angle, and two simple roots form a $3\pi/4$ angle