

MUUTELLUT BESSELIN FUNKTIOT

$I_\nu(z), K_\nu(z)$

TOTEUTTAVAT

TÄMÄN TERMIN ETUMERKKEI VAIHTUNUT

$$\frac{d^2 f}{dz^2} + \frac{1}{z} \frac{df}{dz} - \left(1 + \frac{\nu^2}{z^2}\right) f = 0$$

$z \rightarrow iz$ BESSELIN YHTÄLÖSSÄ $\Rightarrow Z_\nu(iz)$ ON RATKAISU, $Z = J, Y, H_1(z)$

$$I_\nu(z) \equiv e^{-\frac{i\pi\nu}{2}} J_\nu(iz) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{\nu+2k}$$

JOTTA $I_\nu(x)$ OLSI REAALINEN, KUN $x \in \mathbb{R}$

ON $z=0$:SSA SÄÄNNÖLLINEN RATKAISU ($\nu > 0$)

JOTTA $K_\nu(x) \in \mathbb{R}$ KUN $x \in \mathbb{R}$

$$K_\nu(z) \equiv \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(iz) = \frac{\pi}{2} i^{\nu+1} (J_\nu(iz) + i Y_\nu(iz)) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

"MACDONALDIN FUNKTIO" (HECTOR MACDONALD 1865-1935)

$$z \rightarrow 0 \quad K_0(z) = -\log\left(\frac{z}{2}\right) - \gamma + \dots$$

$$K_\nu(z) = \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu} + \dots$$

HYÖDYLLINEN INTEGRAALIESITYS:

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}} e^{-t - \frac{z^2}{4t}}$$

PALAUTUSKAAVOJA

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k} \quad \Gamma(\nu+k+1) = (\nu+k)! \Gamma(\nu)$$

$$\Rightarrow \frac{d}{dz} (z^\nu J_\nu(z)) = \frac{d}{dz} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{\nu+2k} k! \Gamma(\nu+k+1)} z^{\nu+2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu+k)} \left(\frac{z}{2}\right)^{\nu+2k-1} \cdot z^\nu = z^\nu J_{\nu-1}(z)$$

$$\Rightarrow \frac{1}{z} \frac{d}{dz} (z^\nu J_\nu(z)) = z^{\nu-1} J_{\nu-1}(z) \quad (1)$$

TOISTETAAN M KERTAA \Rightarrow

$$\left(\frac{1}{z} \frac{d}{dz}\right)^m (z^\nu J_\nu(z)) = z^{\nu-m} J_{\nu-m}(z) \quad (2) \quad m=1, 2, \dots$$

VASTAANVASTI

$$\frac{d}{dz} (z^{-\nu} J_\nu(z)) = \frac{d}{dz} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{\nu+2k} k! \Gamma(\nu+k+1)} z^{-\nu+2k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{\nu+2k-1} (k-1)! \Gamma(\nu+k-1)} z^{2k-1} = z^{2k-1} \quad q = k-1$$

$$= - \sum_{q=0}^{\infty} \frac{(-1)^q}{2^{\nu+2q+1} q! \Gamma(\nu+q+2)} z^{2q+1} \cdot z^{-\nu} = -z^{-\nu} J_{\nu+1}(z)$$

$$= -z^{-\nu} \sum_{q=0}^{\infty} \frac{(-1)^q}{q! \Gamma(\nu+q+1)} \left(\frac{z}{2}\right)^{2q+\nu+1} = -z^{-\nu} J_{\nu+1}(z)$$

$$\Rightarrow \frac{1}{z} \frac{d}{dz} (z^{-\nu} J_\nu(z)) = -z^{-(\nu+1)} J_{\nu+1}(z) \quad (3)$$

TOISTETAAN M KERTAA \Rightarrow

$$\left(\frac{1}{z} \frac{d}{dz}\right)^m (z^{-\nu} J_\nu(z)) = (-1)^m z^{-(\nu+m)} J_{\nu+m}(z) \quad (4)$$

(1) SAJON $z^v J'_v + v z^{v-1} J_v = z^v J_{v-1}$

ELI $z J'_v + v J_v - z J_{v-1} = 0$ (5)

(2) SAJON $z^{-v} J'_v - v z^{-v-1} J_v = -z^{-v} J_{v+1}$

ELI $z J'_v - v J_v + z J_{v+1} = 0$ (6)

(5) + (6) $\Rightarrow 2z J'_v + z (J_{v+1} - J_{v-1}) = 0$ ELI

$J_{v-1}(z) - J_{v+1}(z) = 2 J'_v(z)$

CPAINOVINNE
C-ROUSTRÖKISÖS

(5) - (6) $\Rightarrow 2v J_v = z (J_{v+1} + J_{v-1}) = 0$ ELI

$J_{v+1}(z) + J_{v-1}(z) = \frac{2v}{z} J_v(z)$

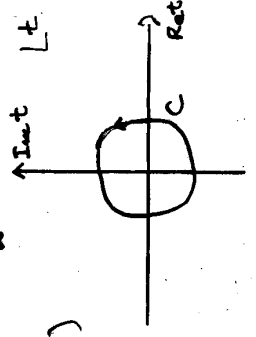
MUODOSTAJAFUNKTIO

$G(z,t) \equiv e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$

TODISTUS:

KIRJOITAMME $e^{\frac{z}{2}(t-\frac{1}{t})} = G(z,t) = \sum_{n=-\infty}^{\infty} a_n(z) t^n$

$\Rightarrow a_n(z) = \frac{1}{2\pi i} \oint_C dt t^{-n-1} G(z,t)$



C: OSAKOKESKEINEN YMPYRÄ
VÄHÄN VÄHÄN

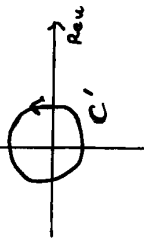
$a_n(z) = \frac{1}{2\pi i} \oint_{C'} \frac{dt}{t^{n+1}} e^{\frac{z}{2}(t-\frac{1}{t})}$

MUUTTUNAN VAIHDOSS: $t \rightarrow u = \frac{z t}{2}$ (OL. $z \neq 0$)

$\frac{dt}{t^{n+1}} = \left(\frac{z}{2}\right)^n \frac{du}{u^{n+1}}$ $|u| = \left|\frac{z}{2}\right| |t|$

$\Rightarrow a_n(z) = \left(\frac{z}{2}\right)^n \frac{1}{2\pi i} \oint_{C'} \frac{du}{u^{n+1}} e^u e^{-\frac{z^2}{4u}}$

$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{z}{2}\right)^{n+2r} \frac{1}{2\pi i} \oint_{C'} du u^{-(r+n+1)} e^u$



NYT $\oint_{C'} du u^{-(r+n+1)} e^u = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{2\pi i} \oint_{C'} du u^{-r-n-1+k}$

$= \frac{1}{(r+n)!} = \frac{1}{\Gamma(r+n+1)}$

SIIS

$a_n(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(r+n+1)} \left(\frac{z}{2}\right)^{2r+n} = J_n(z)$

$n=0, \pm 1, \pm 2, \dots$
□

$z=0: G(0,t) = 1 = \sum_{n=-\infty}^{\infty} a_n(0) t^n$

$\Rightarrow a_0(0) = 1$
 $a_n(0) = 0 \quad n \neq 0$ } PÄTEE MÖKS $J_n(0)$ ILLE

SEURAAVAINA:

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GEN. FUNKTIO, $G(x,t) = e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$

JOS $t = e^{i\varphi}$ $t - \frac{1}{t} = e^{i\varphi} - e^{-i\varphi} = 2i \sin \varphi$

$\Rightarrow e^{ix \sin \varphi} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\varphi}$ (F)

(F) ON FOURIER'N SARJA, KERROINTEN KAAVASTA

$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi e^{i(x \sin \varphi - n\varphi)}$ $n \in \mathbb{Z}$

(INTEGROIMALLI SAMALLA YLI ALIÄN TAVALLA 2π -PIIVUISEN VÄLILLÄ)

KUN $x \in \mathbb{R}$, $I_{-n} J_n(x) = 0$

$\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \sin(x \sin \varphi - \varphi) = \int_{-\pi}^{\pi} \text{pariton funktio}$
 $= 0$ OK.

$\Rightarrow J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \cos(x \sin \varphi - n\varphi)$
 $= \frac{1}{\pi} \int_0^{\pi} d\varphi \cos(x \sin \varphi - n\varphi)$

TOINEN EKUIVALENTTI MUOTO

KUN $x \in \mathbb{R}$ KAIVAN (F) REALIOSA ON $(-1)^n J_n$
 $\cos(x \sin \varphi) = \sum_{n=-\infty}^{\infty} J_n(x) \cos n\varphi = J_0(x) + \sum_{n=1}^{\infty} (J_n(x) + J_{-n}(x)) \cos n\varphi$
 $= J_0(x) + 2 \sum_{n=1}^{\infty} J_n(x) \cos(2n\varphi)$

IMAGINAARIOSA TANS ANTANA

$\sin(x \sin \varphi) = \sum_{n=-\infty}^{\infty} J_n(x) \sin n\varphi$
 $= \sum_{n=1}^{\infty} (J_n(x) - (-1)^n J_n(x)) \sin n\varphi$
 $= 2 \sum_{n=1}^{\infty} J_{2n+1}(x) \sin(2n+1)\varphi$

3) "YHTEENLASKUTEOREEMA"

$G(x+y, t) = G(x, t)G(y, t) = \sum_{p=-\infty}^{\infty} J_p(x)t^p \sum_{q=-\infty}^{\infty} J_q(y)t^q$
 $\sum_{n=-\infty}^{\infty} J_n(x+y)t^n = \sum_{n=-\infty}^{\infty} \left(\sum_{p=-\infty}^{\infty} J_p(x) J_{n-p}(y) \right) t^n$

$\Rightarrow J_n(x+y) = \sum_{p=-\infty}^{\infty} J_p(x) J_{n-p}(y)$

ERKOISETI JOS $n=0$ JA $x=-y$

$J_0(0) = 1 = \sum_{p=-\infty}^{\infty} J_p(x) J_{-p}(-x) = \sum_{p=-\infty}^{\infty} J_p^2(x)$
 $(-1)^p J_p(-x) = J_p(x)$

$\Leftrightarrow J_0^2(x) + 2 \sum_{p=1}^{\infty} J_p^2(x) = 1$

$\Rightarrow |J_0(x)| \leq 1; |J_p(x)| \leq \frac{1}{\sqrt{2}} \quad p=1, 2, \dots$

INTEGRAALIESITYKSIÄ

$$G(x, t) = e^{\frac{x}{2}(t - \frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

JOHDETTIIN:

$$J_n(x) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{dt}{t^{n+1}} e^{\frac{x}{2}(t - \frac{1}{t})} \quad n = 0, \pm 1, \pm 2, \dots$$

"SCHLÄFLIN INTEGRAALI" (LUDWIG SCHLÄFLI 1814-1895)

POISSONIN INTEGRAALIESITYS: (SIMÉON-DEVIS POISSON) (1781-1842)

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_{-1}^1 du e^{iuz} (1-u^2)^{\nu - \frac{1}{2}}$$

Re $\nu > -\frac{1}{2}$

JOHTO: KATSO CROONSTRÖM

KÄYTTÄYTYMINEN PIENILLÄ JA SUURILLA ARGUMENTIN ARVOILLA

SARJASTA: $J_\nu(x) = \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu (1 + O(x^2))$ $x \rightarrow 0$

(TÄMÄ TULOS KÄYTTÄYTYMINEN ∞) $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$ $\nu \neq -1, -2, \dots$

$\rightarrow_{x \rightarrow 0} \frac{(-1)^n}{n!} \left(\frac{x}{2}\right)^n (1 + O(x^2))$

$x \in \mathbb{R}, |x| \rightarrow \infty$

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left(\cos\left(x - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right) + O\left(\frac{1}{x}\right) \right)$$

TODISTUS: KATSO KIRJALLISUUS (ESIM. CROONSTRÖM PÄIKÄLÄ 4.2.9)

SEURAAVA: FUNKTIOLLA $J_\nu(x)$ ON ÄÄRETTÖMÄN MOUNTA NOLLA KOHTAA REAALIASELILLÄ

ORTOGONAALITEORI

FUNKTIOT $u(x) = J_\nu(ax)$ JA $v(x) = J_\nu(bx)$ TOTEUTTAVAT

$$u'' + \frac{1}{x}u' + (a^2 - \frac{\nu^2}{x^2})u = 0 \quad \left| \begin{array}{l} \cdot x u \\ \cdot x u \end{array} \right. \quad \begin{array}{l} \text{VÄHEN-} \\ \text{NETÄÄN} \\ \text{TOINEN-} \\ \text{TAMU} \end{array}$$

$$v'' + \frac{1}{x}v' + (b^2 - \frac{\nu^2}{x^2})v = 0$$

$$\Rightarrow x(\nu u'' - \nu v'') + (\nu u' - \nu v') + (a^2 - b^2)x \nu v = 0$$

$$= \frac{d}{dx} [x(\nu u' - \nu v')]]$$

INTEGROIDAAN $\int_0^R dx$:

$$\Rightarrow (b^2 - a^2) \int_0^R dx \times J_\nu(ax) J_\nu(bx) = \int_0^R x [a J_\nu'(ax) J_\nu(bx) - b J_\nu(ax) J_\nu'(bx)] \quad (1)$$

PALAUTUSKAAVOISTA SAADAAAN (OL. ENSIN $\nu \neq 0$)

$$J_\nu(ax) = \frac{ax}{2\nu} (J_{\nu-1}(ax) + J_{\nu+1}(ax)) \quad a = \frac{a}{b} \text{ TAI } b$$

$$J_\nu'(ax) = \frac{1}{2} (J_{\nu-1}(ax) - J_{\nu+1}(ax)) \quad -u -$$

$$\Rightarrow x [a J_\nu'(ax) J_\nu(bx) - b J_\nu(ax) J_\nu'(bx)]$$

$$= \frac{ax}{2} [J_{\nu-1}(ax) J_{\nu+1}(bx) - J_{\nu+1}(ax) J_{\nu-1}(bx)]$$

$$\xrightarrow{R \rightarrow 0} \rightarrow 0 \text{ KUN } x \rightarrow 0 \text{ JOS } \nu > -1$$

$$\nu = 0: J_0'(ax) = 1 + k(ax)^2 + \dots$$

$$J_0'(bx) = 2kax + \dots$$

$$\Rightarrow x [a J_0'(ax) J_0(bx) - b J_0(ax) J_0'(bx)] = \text{vakio} \cdot x^2 \rightarrow 0$$

$$x \rightarrow 0$$

VALITTAAN NYT a, b S.E. SIVOITUSTERMI = 0 MYÖS YLÄRAJALLA, ESIM.

$$J_\nu(aR) = J_\nu(bR) = 0$$

OLKoon $x_{0i} J_\nu(x)$:N IIS NOLLAKOHTA KUN $x > 0$ (NOLLAKOHTIA OLI OEBIN MONTA)

$$\text{VAL. } a = \frac{x_{0i}}{R}, \quad b = \frac{x_{0j}}{R}$$

$$\Rightarrow \frac{1}{R^2} (x_{0j}^2 - x_{0i}^2) \int_0^R dx \times J_\nu(\frac{x_{0i}}{R}x) J_\nu(\frac{x_{0j}}{R}x) = 0$$

JOS $x_{0i} \neq x_{0j}$

$$\Rightarrow \int_0^R dx \times J_\nu(\frac{x_{0i}}{R}x) J_\nu(\frac{x_{0j}}{R}x) = 0 \quad \nu > -1$$

MITÄ JOS $x_{0i} = x_{0j}$?

KAAVASSA (1) VALITTAAN $a = \frac{x_{0i}}{R}, \quad b = \frac{x_{0i}}{R} + \epsilon$

$$b^2 - a^2 = (b+a)(b-a) = 2\frac{x_{0i}}{R} \cdot \epsilon + O(\epsilon^2)$$

$$\Rightarrow \int_0^R dx \times [J_\nu(\frac{x_{0i}}{R}x)]^2 = \lim_{\epsilon \rightarrow 0} \frac{1}{2\frac{x_{0i}}{R} \cdot \epsilon} \cdot R \left[\frac{x_{0i}}{R} J_\nu'(\frac{x_{0i}}{R}x) J_\nu(\frac{x_{0i}}{R}x) - (\frac{x_{0i}}{R} + \epsilon) J_\nu(\frac{x_{0i}}{R}x) J_\nu'(\frac{x_{0i}}{R}x) \right]$$

$$= \frac{R}{2} J_\nu'(\frac{x_{0i}}{R}x) \cdot \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} J_\nu(\frac{x_{0i}}{R}x + \epsilon) = \frac{R^2}{2} [J_\nu'(\frac{x_{0i}}{R}x)]^2$$

PALAUTUSKAAVAT:

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_{\nu}(x)$$

LASKEMALLA YHTEEN $\Rightarrow J'_{\nu}(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_{\nu}(x)$

VÄHENTÄMÄLLÄ $\Rightarrow J'_{\nu}(x) = \frac{\nu}{x} J_{\nu}(x) - J_{\nu+1}(x)$

$$\Rightarrow J'_{\nu}(x_{0i}) = J_{\nu-1}(x_{0i}) - J_{\nu+1}(x_{0i})$$

$$\Rightarrow \int_0^R dx \times J_{\nu}\left(\frac{x_{0i}}{R}x\right) J_{\nu}\left(\frac{x_{0j}}{R}x\right) = \frac{R^2}{2} [J_{\nu \pm 1}(x_{0i})]^2 \delta_{ij} \quad \nu > 1$$

PALLOBESSELIT

HELMHOLTZIN YHTÄLÖN $(\nabla^2 + k^2)u = 0$ SEPAROINTI
PALLOKOORDINAATEISSA JOHTAA RADIAALISEEN
YHTÄLÖÖN

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + [k^2 r^2 - \ell(\ell+1)]R = 0$$

SIRREYTÄÄ MUUTTUVAAN $x = kr$

$$x^2 \frac{d^2 R}{dx^2} + 2x \frac{dR}{dx} + [x^2 - \ell(\ell+1)]R = 0$$

YRITE: $R(x) = \frac{Z(x)}{\sqrt{x}}$

$$\Rightarrow x^2 R'' + 2x R' + [x^2 - \ell(\ell+1)]R =$$

$$= \frac{1}{\sqrt{x}} (x^2 Z'' + x Z' + (x^2 - (\ell + \frac{1}{2})^2) Z) = 0$$

$\Rightarrow Z$ SYLINDERIFUNKTIONIN MURE... 0 < 1

PALLO BESSSEL $J_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$

PALLO NEUMANN $N_n(x) = \sqrt{\frac{\pi}{2x}} N_{n+\frac{1}{2}}(x)$
 $= (-1)^{n+1} \sqrt{\frac{\pi}{2x}} J_{-n-\frac{1}{2}}(x)$

$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{1}{\Gamma(\nu+s+1)} \left(\frac{x}{2}\right)^{2s} \sqrt{\frac{x}{2}}$

$J_{\frac{1}{2}}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{1}{\Gamma(s+\frac{3}{2})} \left(\frac{x}{2}\right)^{2s} \sqrt{\frac{x}{2}}$

NYT $\Gamma(s+\frac{3}{2}) = (s+\frac{1}{2})\Gamma(s+\frac{1}{2}) = \dots = (s+\frac{1}{2})(s-\frac{1}{2}) \dots \frac{1}{2} \Gamma(\frac{1}{2})$
 $= \frac{(2s+1)!!}{2^{s+1}} \sqrt{\pi}$

$(2s+1)!! = 1 \cdot 3 \cdot 5 \dots (2s+1)$
 $\Rightarrow J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{2^{s+1}}{(2s+1)!!} \sqrt{\pi} \frac{x^{2s}}{2^{2s}} = \sqrt{\frac{2x}{\pi}} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{(2s+1)!} \frac{1}{x} = \sqrt{\frac{2x}{\pi}} \frac{\sin x}{x}$

$\Rightarrow J_0(x) = \frac{\sin x}{x}$

PALAUTUSKAAVA $J_{\nu+1} = \frac{\nu}{x} J_\nu - J'_\nu = -x^\nu \frac{d}{dx} (x^{-\nu} J_\nu)$

$\Rightarrow x^{-(\nu+1)} J_{\nu+1} = -\frac{1}{x} \frac{d}{dx} (x^{-\nu} J_\nu) = \left(-\frac{1}{x} \frac{d}{dx}\right)^2 (x^{-\nu} J_\nu)$

$\Rightarrow x^{-(n+1)} J_{n+1}(x) = (-1)^n \left(\frac{1}{x} \frac{d}{dx}\right)^n \left(\frac{1}{\sqrt{x}} J_{\frac{1}{2}}(x)\right)$

$\Rightarrow J_n(x) = x^n (-1)^n \left(\frac{1}{x} \frac{d}{dx}\right)^n \frac{\sin x}{x}$

$\Rightarrow J_n(x) = \frac{\sin x}{x^n} - \frac{\cos x}{x} , j_n(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin x - \frac{3}{x^2} \cos x$

VASTAAVASTI $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2x}{\pi}} \frac{\cos x}{x}$

JOSTA $n_0(x) = -\frac{\cos x}{x}$

PALAUTUSKAAVASTA $J_{\nu-1} = \frac{\nu}{x} J_\nu + J'_\nu$
 $= x^{-\nu} \frac{d}{dx} (x^\nu J_\nu)$

SEURAA $x^{\nu-1} J_{\nu-1}(x) = \frac{1}{x} \frac{d}{dx} (x^\nu J_\nu)$

$\Rightarrow J_{-n-\frac{1}{2}}(x) = \left(\frac{1}{x} \frac{d}{dx}\right)^n \left(\frac{1}{\sqrt{x}} J_{-\frac{1}{2}}(x)\right)$

$\Rightarrow n_n(x) = (-1)^n \left(\frac{1}{x} \frac{d}{dx}\right)^n n_0(x)$

$= (-1)^{n+1} \left(\frac{1}{x} \frac{d}{dx}\right)^n \frac{\cos x}{x}$

$\Rightarrow n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$

$n_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right) \cos x - \frac{3}{x^2} \sin x$

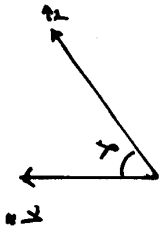
RAJA-ARVOJA: $J_n(x) \approx \frac{x^n}{(2n+1)!!}$
 $x \rightarrow 0$

$n_n(x) \approx -(2n-1)!! x^{-n-1}$ $n \geq 1$

$n_0(x) \approx -\frac{1}{x}$

$|x| \rightarrow \infty$ $J_n(x) \rightarrow \frac{1}{x} \sin\left(x - \frac{n\pi}{2}\right)$

$n_n(x) \rightarrow -\frac{1}{x} \cos\left(x - \frac{n\pi}{2}\right)$

TASOALION PALLOALTOKEHITELMÄ

$$e^{i\mathbf{k} \cdot \mathbf{r}} = e^{i k r \cos \theta}$$

$$= \sum_{l=0}^{\infty} (2l+1) f_l(kr) P_l(\cos \theta)$$

$$f_l(kr) = \frac{1}{2} \int_{-1}^1 d\cos \theta e^{i k r \cos \theta} P_l(\cos \theta)$$

OSOITAMME, ETTÄ $f_l(kr) \propto j_l(kr)$:

POISSONIN INTEGRALILIIKITYS (CRONSTRÖM §4.2.6)

$$\Rightarrow \sqrt{\frac{2\pi}{x}} J_{l+\frac{1}{2}}(x) = \frac{x}{2^l l!} \int_{-1}^1 du e^{iux} (1-u^2)^l$$

OSITTAISINTEGROIDAAN:

$$\int_{-1}^1 du e^{iux} (1-u^2)^l = \frac{i}{x} \int_{-1}^1 du e^{iux} \frac{d}{du} (1-u^2)^l = \dots$$

$$= \left(\frac{i}{x}\right)^l \int_{-1}^1 du e^{iux} \frac{d^l}{du^l} (1-u^2)^l$$

$$\Rightarrow \sqrt{\frac{2\pi}{x}} J_{l+\frac{1}{2}} = (-i)^l \int_{-1}^1 du e^{iux} \underbrace{\frac{1}{2^l l!} \frac{d^l}{du^l} (1-u^2)^l}_{P_l(u) \text{ (RODRIGUES)}} \\ \text{"} \\ 2 j_l(kr)$$

$$\Rightarrow f_l(kr) = i^l j_l(kr)$$

$$\Rightarrow e^{i\mathbf{k} \cdot \mathbf{r}} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta)$$

$$= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l i^l j_l(kr) Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta, \varphi)$$

YHTEENLASKULAUSE

MISSÄ MERKITTIIN $\vec{k} = k (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$

$\vec{r} = r (\sin \theta_r \cos \varphi_r, \sin \theta_r \sin \varphi_r, \cos \theta_r)$

LAGUERREN FUNKTIOT

(EDMOND LAGUERRE 1834-1886)

LAGUERREN YHTÄLÖ :

$$z \frac{d^2 f}{dz^2} + (1-z) \frac{df}{dz} + \mu f(z) = 0$$

↑
PARAMETRI

z=0 HEIKKO ERIKOISPISTE

POTENSIISSARJAYRITE : $f(z) = \sum_{k=0}^{\infty} b_k z^{k+r}$

$z f'' = b_0 r(r-1) z^{r-1} + \dots$

$(1-z) f' = b_0 r z^{r-1} + \dots$

$\mu f = \mu b_0 z^r + \dots$

KARAKTERISTINEN YHTÄLÖ: $r(r-1) + r + \mu = r^2 = 0$

r=0 KAKSOISJUURI => SAADAAN z=0:ssa

SÄÄNNÖLLINEN RATKAISU $f(z) = \sum_{k=0}^{\infty} b_k z^k = \mathcal{L}_\mu(z)$

JÄ TOINEN RATKAISU $\propto \log z, z \neq 0$

r=0 => PALAUTUSKAAVA

$$b_{k+1} = \frac{(k-\mu)}{(k+1)} \frac{b_k}{z}$$

RATKAISU

$$b_k = \frac{b_k}{b_{k-1}} \frac{b_{k-1}}{b_{k-2}} \dots \frac{b_1}{b_0} b_0$$

$$= \frac{(k-1-\mu)}{k^2} \frac{(k-2-\mu)}{(k-1)^2} \dots \frac{(-\mu)}{1^2} b_0$$

$$= \frac{(-\mu)_k}{(k!)^2} b_0$$

KÄYTTÄYTYMINEN ISOILLA |z| :

$$\frac{b_m}{b_{m-1}} = \frac{m-1-\mu}{m^2} \rightarrow \frac{1}{m} \quad m \rightarrow \infty$$

$$b_m \propto \frac{1}{m!} \quad m \rightarrow \infty$$

$$\Rightarrow \mathcal{L}_\mu(z) \sim e^{-az} \quad (a > 0) \quad z \rightarrow +\infty$$

POIKKEUS: JOS $\mu = n$, KOKONAISLUKU

$(-n)_k = 0 \quad k > n \Rightarrow \mathcal{L}_n(z)$ ON POLYNOMI
ASTETTA n

" $\frac{n!}{(n-k)!} \quad (-n)_k = (k-1-n)(k-2-n) \dots (-n) = (-1)^k n \cdot (n-1) \dots (n-k)$

$\Rightarrow b_k = n! \frac{(-1)^k}{(k!)^2 (n-k)!} b_0$

KUN VALITAAN $b_0 = n!$ SAADAAN LAGUERREN POLYNOMIT

$$\mathcal{L}_n(z) = (n!)^2 \sum_{k=0}^n \frac{(-1)^k}{(k!)^2 (n-k)!} z^k = (-1)^n z^n + \dots$$

ESIN.

$\mathcal{L}_0(z) = 1$

$\mathcal{L}_1(z) = 1 - z$

$\mathcal{L}_2(z) = 2 - 4z + z^2$

$\mathcal{L}_3(z) = 6 - 18z + 9z^2 - z^3$ ONE

HUOM! MUITAKIN NORMITUSSOPIMUKSIA LÖYTYY!

LAGUERREN LIITTOPOLYNOMIT

$$L_n^{(p)} \equiv \frac{d^p}{dz^p} L_n(z) \quad p = 0, 1, \dots, n$$

$$L_n^{(0)} = L_n(z) \quad L_n^{(n)}(z) = (-1)^n n! \quad \text{VAKIO}$$

$$L_n^{(p)}(z) = \sum_{k=p}^n \frac{(-1)^k (n!)^2}{k!(n-k)!(k-p)!} z^{k-p}$$

TOTEUTTAVAT MINKÄ YHTÄLÖN ?

DERIVOIDAAN LAGUERREN YHTÄLÖÄ ($\mu = n$)

P KERTAA:

$$0 = \frac{d^p}{dz^p} \left(z \frac{dL_n}{dz} + (1-z) \frac{dL_n}{dz} + nL_n \right) =$$

$$= z \frac{d^2 L_n^{(p)}}{dz^2} + p \frac{d L_n^{(p)}}{dz} + \frac{d L_n^{(p)}}{dz} L_n^{(p)} - z \frac{d L_n^{(p)}}{dz} - p L_n^{(p)} + n L_n^{(p)} =$$

$$= z \frac{d^2 L_n^{(p)}}{dz^2} + (p+1-z) \frac{d L_n^{(p)}}{dz} + (n-p) L_n^{(p)} = 0$$

LAGUERREN LIITTYYHTÄLÖ

ORTONORMITUS

TODISTETAAN ENSIN RODRIGUES-TYYPINEN

KAAVA

$$L_n(z) = e^z \frac{d^n}{dz^n} (z^n e^{-z}) \quad (LR)$$

$$\text{ON: } \frac{d^n}{dz^n} (z^n e^{-z}) = \sum_{k=0}^n \binom{n}{k} \frac{d^k e^{-z}}{dz^k} \underbrace{\frac{d^{n-k} z^n}{dz^{n-k}}}_{\frac{n!}{k!} z^k} (-1)^k e^{-z} = \frac{n!}{k!(n-k)!} (-1)^k (n!)^2 z^k = e^{-z} L_n(z)$$

OSOITETAAN SEURAAVAKSI ETTÄ LAGUERREN

POLYNOMIT L_n OVAT ORTOGONAALISIA POLYNOMEJA VÄLILLÄ $[0, \infty)$ PAINOFUNKTIOLLA $w(x) = e^{-x}$, S.O.

$$\int_0^{\infty} dx e^{-x} L_m(x) L_n(x) \propto \delta_{mn}$$

TARKASTELETAAN ($n \geq m$)

$$I_{mn} = \int_0^{\infty} dx e^{-x} L_m(x) L_n(x) =$$

$$= \int_0^{\infty} dx \left(\frac{d^n}{dx^n} (x^n e^{-x}) \right) L_m(x) =$$

$$= \int_0^{\infty} \underbrace{\left(\frac{d^{n-1}}{dx^{n-1}} x^n e^{-x} \right)}_{-n} L_m(x) - \int_0^{\infty} dx \left(\frac{d^{n-1}}{dx^{n-1}} x^n e^{-x} \right) \frac{dL_m}{dx}$$

$$= \dots = (-1)^n \int_0^{\infty} dx x^n e^{-x} \frac{d^m \mathcal{L}_m}{dx^m} = 0 \text{ JOS } n > m$$

$$\Rightarrow I_{mn} = 0 \quad m \neq n$$

$$\text{KUN } m = n \quad \frac{d^n \mathcal{L}_n}{dx^n} = \frac{d^n}{dx^n} (-1)^n x^n = (-1)^n n!$$

$$\Rightarrow I_{nn} = n! \int_0^{\infty} dx x^n e^{-x} = (n!)^2$$

$$\Rightarrow \int_0^{\infty} dx e^{-x} \mathcal{L}_m(x) \mathcal{L}_n(x) = (n!)^2 \delta_{mn}$$

LAGUERREN FUNKTIOT $F_n(x) = e^{-x/2} \mathcal{L}_n(x)$ OVAT
SIIS ORTOGONAALISIA VÄLILLÄ $[0, \infty)$:

$$\int_0^{\infty} dx F_n(x) F_m(x) = (n!)^2 \delta_{mn}$$

MUODOSTAJAFUNKTIO

FYMM I $f(z)$ ANALYTTINEN $\Rightarrow \frac{d^n f}{dz^n}(z) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{(\xi-z)^{n+1}}$

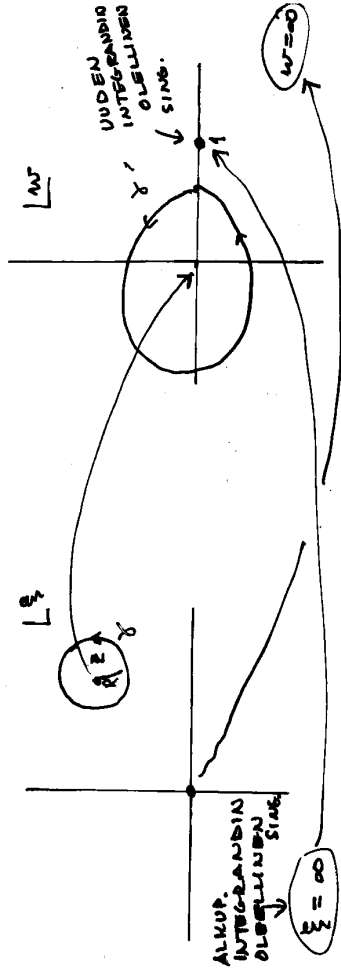
OTETAAN $f(z) = z^n e^{-z}$

$$(LR) \Rightarrow e^{-z} \mathcal{L}_n(z) = \frac{n!}{2\pi i} \oint_{\gamma} \xi^n e^{-\xi} \frac{1}{(\xi-z)^{n+1}}$$

$$\frac{1}{n!} \mathcal{L}_n(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\xi}{\xi} \left(\frac{\xi}{\xi-z} \right)^{n+1} e^{-(\xi-z)}$$

UUSI MUUTTUJA $w = \frac{\xi-z}{\xi} \Leftrightarrow \xi = \frac{z}{1-w}$

$$\frac{d\xi}{\xi} \left(\frac{\xi}{\xi-z} \right)^{n+1} e^{-(\xi-z)} = \frac{dw}{w^{n+1}} e^{-\frac{zw}{1-w}}$$



ESIM. $\xi - z = Re^{i\theta} \quad \frac{R}{|z|} < 1, \quad z = |z| e^{+i\alpha}$

$$\Rightarrow \gamma': \quad w = 1 + \frac{|z|}{R} e^{i\varphi} \quad \varphi = \theta - \alpha$$

γ' KIERTÄÄ $w=0, \infty$ KERRAN VASTAPÄIVÄÄN

$$\Rightarrow \frac{1}{n!} \mathcal{L}_n(z) = \frac{1}{2\pi i} \oint_{\gamma'} \frac{dw}{w^{n+1}} e^{-\frac{zw}{1-w}} = \frac{1}{n!} \frac{\partial^n}{\partial w^n} \left(e^{-\frac{zw}{1-w}} \right)$$

$$\Rightarrow \mathcal{L}_0(z, w) = \frac{e^{-\frac{zw}{1-w}}}{1-w} = \sum_{n=0}^{\infty} \frac{\mathcal{L}_n(z)}{n!} w^n$$

LITTOPOLYNOMIEN MUODOSTAJA -

PUNKTIO

$$\mathcal{L}_n^{(k)}(z) = \frac{d^k}{dz^k} \mathcal{L}_n(z) \quad ; \quad G_0(z, w) = \frac{e^{-\frac{zw}{1-w}}}{1-w}$$

LASKETAAN $\frac{\partial^k}{\partial z^k} G_0(z, w) = \sum_{n=k}^{\infty} \frac{1}{n!} \mathcal{L}_n^{(k)}(z) w^n$

" $\frac{(-1)^k w^k}{(1-w)^{k+1}} e^{-\frac{zw}{1-w}}$

NÄKETEELLÄÄN $G_k(z, w) = \frac{(-1)^k}{(1-w)^{k+1}} e^{-\frac{zw}{1-w}}$

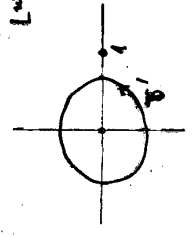
JÄ SAADAAN (m = n - k)

$$G_k(z, w) = \sum_{m=0}^{\infty} \frac{1}{(m+k)!} \mathcal{L}_{m+k}^{(k)}(z) w^m$$

JUHDETAAN TÄSTÄ "RODRIGUES-KAAVA" LITTOPOLY-NOMEILLE:

$$\frac{1}{(m+k)!} \mathcal{L}_{m+k}^{(k)}(z) = \frac{1}{m!} \frac{\partial^m}{\partial w^m} G_k(z, w) \Big|_{w=0}$$

$$L^w = \frac{1}{m!} \oint \frac{dw}{w^{m+1}} \frac{(-1)^k}{(1-w)^{k+1}} e^{-\frac{zw}{1-w}}$$



PALAUTUSKAAVOJA:

1) $\frac{\partial}{\partial w} \left(\frac{e^{-\frac{zw}{1-w}}}{1-w} \right) = \frac{e^{-\frac{zw}{1-w}}}{1-w} \frac{1-w-z}{(1-w)^2} \quad ; \quad ELI$

$$(1-w)^2 \frac{\partial}{\partial w} G_0(z, w) = (1-w-z) G_0(z, w) \sum_{n=0}^{\infty} \frac{\mathcal{L}_n(z)}{n!} w^n$$

WÄ: KERRDIN MOLEKULIN PUOLIN:

$$\frac{1}{n!} \mathcal{L}_{n+1} - 2 \frac{\mathcal{L}_n}{(n-1)!} + \frac{\mathcal{L}_{n-1}}{(n-2)!} = (1-z) \frac{\mathcal{L}_n}{n!} - \frac{\mathcal{L}_{n-1}}{(n-1)!} \quad | \cdot n!$$

$$\Rightarrow \mathcal{L}_{n+1}(z) - 2n \mathcal{L}_n(z) + n(n-1) \mathcal{L}_{n-1}(z) = (1-z) \mathcal{L}_n(z) - n \mathcal{L}_{n-1}(z)$$

$$\mathcal{L}_{n+1}(z) - (2n+1-z) \mathcal{L}_n(z) + n^2 \mathcal{L}_{n-1}(z) = 0$$

2) $\frac{\partial}{\partial z} \left(\frac{e^{-\frac{zw}{1-w}}}{1-w} \right) = -\frac{w}{1-w} \frac{e^{-\frac{zw}{1-w}}}{1-w} \quad ; \quad ELI$

$$(1-w) \frac{\partial}{\partial z} G_0(z, w) = -w \sum_{n=0}^{\infty} \frac{\mathcal{L}_n(z)}{n!} w^n$$

$$\Rightarrow \frac{\mathcal{L}'_n}{n!} - \frac{\mathcal{L}'_{n-1}}{(n-1)!} = -\frac{\mathcal{L}'_{n-1}}{(n-1)!}$$

$$\Rightarrow \mathcal{L}'_n(z) - n \mathcal{L}'_{n-1}(z) + n \mathcal{L}'_{n-1}(z) = 0$$

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MUUTTUJAN VAIHTOS $w \rightarrow \xi = \frac{r}{a_0} \Leftrightarrow w = \frac{\xi - z}{\xi}$
 (KÄÄNTEINEN ÄSKELLELLE!)

$$\Rightarrow \frac{1}{(m+k)!} \mathcal{L}_{m+k}^{(k)}(z) = \frac{(-1)^k z^{-k} e^{-z}}{2\pi i} \oint_{\gamma} d\xi \frac{\xi^{m+k} e^{-\xi}}{(\xi - z)^{m+1}}$$

$$\stackrel{L_2}{=} \frac{1}{2\pi i} \int_{\gamma} (-1)^k z^{-k} e^{-z} \frac{1}{m!} \frac{d^m}{dz^m} (z^{m+k} e^{-z})$$

$$\Rightarrow \mathcal{L}_{m+k}^{(k)}(z) = (-1)^k \frac{(m+k)!}{m!} z^{-k} e^{-z} \frac{d^m}{dz^m} (z^{m+k} e^{-z})$$

TÄMÄN AVULLA LASKETAAN

$$J_{mn}^{(k)} = \int_0^{\infty} dx e^{-x} x^k \mathcal{L}_{m+k}^{(k)}(x) \mathcal{L}_{n+k}^{(k)}(x)$$

OLKOAAN $m \geq n$

$$J_{mn}^{(k)} = (-1)^k \frac{(m+k)!}{m!} \int_0^{\infty} dx \left(\frac{d^m}{dx^m} (x^{m+k} e^{-x}) \right) \mathcal{L}_{n+k}^{(k)}(x)$$

$$= (-1)^k \frac{(m+k)!}{m!} \int_0^{\infty} \frac{d^{m-1}}{dx^{m-1}} (x^{m+k} e^{-x}) \mathcal{L}_{n+k}^{(k)}(x) \leftarrow = 0$$

$$+ (-1)^{k+1} \frac{(m+k)!}{m!} \int_0^{\infty} dx \frac{d^{m-1}}{dx^{m-1}} (x^{m+k} e^{-x}) \frac{d \mathcal{L}_{n+k}^{(k)}}{dx}$$

$$= \dots = (-1)^{k+m} \frac{(m+k)!}{m!} \int_0^{\infty} dx x^{m+k} e^{-x} \frac{d^m \mathcal{L}_{n+k}^{(k)}}{dx^m}$$

JOS $m > n$, $\frac{d^m \mathcal{L}_{n+k}^{(k)}}{dx^m} = 0$

$$m \neq n \quad \frac{d^m \mathcal{L}_{n+k}^{(k)}}{dx^m} = \frac{d^{n+k}}{dx^{n+k}} (-1)^{n+k} x^{n+k} = (-1)^{n+k} (n+k)!$$

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$$\Rightarrow J_{nn}^{(k)} = (-1)^{2(n+k)} \frac{(n+k)!}{n!} \int_0^{\infty} dx x^{n+k} e^{-x}$$

$$= \frac{[(n+k)!]^2}{n!}$$

SIIS

$$\int_0^{\infty} dx e^{-x} x^k \mathcal{L}_{m+k}^{(k)}(x) \mathcal{L}_{n+k}^{(k)}(x) = \frac{[(n+k)!]^2}{n!} \delta_{mn}$$

LAGUERREN LIITTOPOLYNOMIT (Ovat ORTOGONAALISIA VÄLILLÄ [0,∞) PAINOFUNKTIOLLA

$$w(x) = x^k e^{-x}$$

VETYATOMIN RADIAALINEN YHTÄLÖ

SCHRÖDINGER:

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi + V\psi = E\psi$$

μ HIUKKASEN MASSA
 (TÄSSÄ ELEKTRONIN)

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

YRITE $\psi(r, \theta, \varphi) = R(r) Y_{lm}(\theta, \varphi)$

\Rightarrow RADIAALINEN YHTÄLÖ

$$-\frac{\hbar^2}{2\mu} \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) R + \frac{\hbar^2}{2\mu r^2} l(l+1)R - \frac{e^2}{4\pi\epsilon_0 r} R = ER$$

KIRJ. $R(r) = \frac{u(r)}{r}$

$$\Rightarrow \frac{d^2 u}{dr^2} + \left[-\frac{l(l+1)}{r^2} + \frac{2\mu}{\hbar^2} \frac{e^2}{4\pi\epsilon_0 r} + \frac{2\mu E}{\hbar^2} \right] u = 0$$

MÄÄR. $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx 137,04$ JA DIMENSIOTONNAT SUUREET
 HIENDRAKEVAIKU

$\xi = \frac{\mu c}{\hbar} r \equiv \frac{r}{\lambda_c}$ $\lambda_c = \frac{h}{\mu c} \approx 2,426 \times 10^{-12} \text{ m}$ ELEKTRONIN COMPTON-VAIKU
 $e = \frac{2E}{\mu c^2}$

$$\Rightarrow \frac{d^2 u}{d\xi^2} + \left(\epsilon + \frac{2\alpha}{\xi} - \frac{l(l+1)}{\xi^2} \right) u = 0$$

VIELÄ: $\xi = 2\sqrt{\epsilon} \frac{\xi}{2}$ ($\epsilon < 0$ SIDOTUILLE TILOILLE)

JA $u(\xi) = C \rho^{l+1} e^{-\rho/2} f(\rho)$

$\Rightarrow f$ TOTEUTTAA

$$\rho \frac{d^2 f}{d\rho^2} + (2l+2-\rho) \frac{df}{d\rho} + \left(\frac{\alpha}{\sqrt{\epsilon}} - l-1 \right) f = 0$$

LAGUERREN LIITTOYHTÄLÖ $\rho = 2l+1$
 $n = \frac{\alpha}{\sqrt{\epsilon}} + l$

PÄÄKVAANTILILUKU $N = n-l$

$$E_N = -\frac{\alpha^2}{N^2} \Rightarrow E_N = -\frac{1}{2} \mu c^2 \frac{\alpha^2}{N^2}$$

HERMITEN POLYNOMIT

(CHARLES HERMITE 1822-1901)

MUODOSTAJAFUNKTIO: HERMITEN POLYNOMIT

$$g(x, t) = e^{-t^2 + 2xt} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

$$H_0(x) = g(x, 0) = 1$$

$$H_1(x) t \approx 2xt \Rightarrow H_1(x) = 2x$$

PALAUTUSKAAVOJA:

$$\frac{\partial}{\partial x} g(x, t) = 2t g(x, t) = 2 \sum_{n=1}^{\infty} n H_{n-1} \frac{t^n}{n!}$$

$$\Rightarrow \sum_{n=0}^{\infty} H'_n(x) \frac{t^n}{n!} = 2 \sum_{n=1}^{\infty} n H_{n-1} \frac{t^n}{n!}$$

(PI)

$$g(x,t) = e^{-t^2 + 2xt} = e^{-(t-x)^2} \cdot e^{-x^2}$$

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= 2(x-t)g(x,t) = 2 \sum_{n=0}^{\infty} (x H_n(x) - n H_{n-1}(x)) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} H_{n+1}(x) \frac{t^n}{n!} \end{aligned}$$

$$\Rightarrow \boxed{H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)} \quad (P2)$$

$$\begin{aligned} \Rightarrow H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \end{aligned} \quad \text{JNE.}$$

TOINEN ESITYS:

$$\begin{aligned} (G) \Rightarrow H_n(x) &= \frac{\partial^n}{\partial t^n} g(x,t) \Big|_{t=0} = e^{x^2} \frac{\partial^n}{\partial t^n} e^{-(tx)^2} \Big|_{t=0} \\ &= e^{x^2} \frac{\partial^n}{\partial (-x)^n} e^{-(t-x)^2} \Big|_{t=0} = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \end{aligned}$$

$$\Rightarrow \boxed{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}} \quad (\text{RODRIGUES})$$

DIFFERENTIALIYHTÄLÖ:

$$\begin{aligned} H_n'' &\stackrel{(P1)}{=} 2n H_{n-1}' \stackrel{(P2)}{=} \frac{d}{dx} (2x H_n - H_{n+1}) = 2H_n + 2x H_n' - H_{n+1}' \\ &\stackrel{(P3)}{=} 2H_n + 2x H_n' - 2(n+1)H_n = 2x H_n' - 2n H_n \end{aligned}$$

$$\text{SIIS: } \boxed{\frac{d^2 H_n}{dx^2} - 2x \frac{dH_n}{dx} + 2n H_n = 0}$$

INTEGRAALIKAAVA

OLKON $\Pi_m(x) = A_m x^m + A_{m-1} x^{m-1} + \dots + A_0$ POLYNOMI ASTETTA M

HALUAMME LASKEA

$$I_{mn} = \int_{-\infty}^{\infty} dx e^{-x^2} \Pi_m(x) H_n(x) \quad \text{KUN } m \leq n$$

KÄYTETÄÄN "RODRIGUES-MAANNA" $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

$$\Rightarrow I_{mn} = (-1)^n \int_{-\infty}^{\infty} dx \Pi_m(x) \frac{d^n}{dx^n} e^{-x^2} = (-1)^n \int_{-\infty}^{\infty} \frac{\Pi_m(x)}{\Pi_m(x)} \underbrace{\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}}_{\text{POLYNOMI} \times e^{-x^2}} dx$$

$$\begin{aligned} &+ (-1)^{n+1} \int_{-\infty}^{\infty} dx \frac{d\Pi_m}{dx} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} = \text{SIBOITUSTERMI} = 0 \\ &= \dots = (-1)^{2n} \int_{-\infty}^{\infty} dx \frac{d^n \Pi_m}{dx^n} e^{-x^2} \end{aligned}$$

JOS $m < n$ $\frac{d^n \Pi_m}{dx^n} = 0$, JA $I_{mn} = 0$

JOS $m = n$ $\frac{d^n \Pi_m}{dx^n} = A_n n!$, $\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}$

$$\rightarrow I_{nn} = A_n n! \sqrt{\pi}$$

ERIKOISESTI, JOS $\Pi_m(x) = H_m(x)$:

"RODRIGUES" $H_n(x) = (-1)^n (-2x)^n + \text{ALEMMAN KERTALUVU TERMEJÄ}$
 $\Rightarrow A_n = 2^n$

$$\boxed{\int_{-\infty}^{\infty} dx e^{-x^2} H_m(x) H_n(x) = 2^n n! \sqrt{\pi} \delta_{mn}}$$

HERMITEN FUNKTIOT $\varphi_n(x) = e^{-x^2/2} H_n(x)$

TOTEUTTAVAT :

* DIFFERENTIAALIYHTÄLÖN

$$\frac{d^2 \varphi_n}{dx^2} + (2n+1 - x^2) \varphi_n = 0$$

$$(H_n = e^{x^2/2} \varphi_n \Rightarrow H_n' = e^{x^2/2} (\varphi_n' + x \varphi_n)$$

$$H_n'' = e^{x^2/2} (\varphi_n'' + 2x \varphi_n' + x^2 \varphi_n + \varphi_n)$$

$$\Rightarrow H_n'' - 2x H_n' + 2n H_n = e^{x^2/2} (\varphi_n'' + 2x \varphi_n' + x^2 \varphi_n + \varphi_n - 2x \varphi_n' - 2x^2 \varphi_n + 2n \varphi_n) = e^{x^2/2} (\varphi_n'' + (2n+1-x^2) \varphi_n) = 0$$

* ORTOGONAALISUUSEHDON

$$\int_{-\infty}^{\infty} dx \varphi_m(x) \varphi_n(x) = \sqrt{\pi} 2^n n! \delta_{mn}$$

HERMITEN FUNKTIOT OVAT KVANTIMEKAANISEN HARMONISEN OSKILLAATTORIN ENERGIAN

OMINAISFUNKTIOT :

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2 \quad (\hat{p} = -i\hbar \frac{d}{dx})$$

AVASTA RIIPPUMATON SCHRÖDINGERIN YHTÄLÖ:

$$\hat{H} \psi = -\hbar^2 \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega^2 \psi = E \psi$$

$$\psi = \int_{-\infty}^{\infty} d\omega |\psi(\omega)\rangle^2 < \infty$$

SIIRTYÄN DIMENSIOTONNAAN MUUTTUJAA

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x$$

$$[k] = \frac{ML^2}{T}, \quad [\omega] = \frac{1}{T}$$

$$\Rightarrow \left[\frac{m\omega}{\hbar} \right] = \frac{M}{T} \frac{T}{ML^2} = \frac{1}{L^2} \Rightarrow \left[\sqrt{\frac{m\omega}{\hbar}} x \right] = 1 \quad OK$$

$$\frac{d^2}{dx^2} = \frac{m\omega}{\hbar} \frac{d^2}{d\xi^2}$$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi = \frac{\hbar \omega}{2} \left(-\frac{d^2 \psi}{d\xi^2} + \xi^2 \psi \right) = E \psi$$

MERK. $\lambda = \frac{2E}{\hbar \omega}$

VEK $\varphi_n'' + (2n+1 - \xi^2) \varphi_n = 0$

$$\Rightarrow \frac{d^2 \psi}{d\xi^2} + (\lambda - \xi^2) \psi = 0$$

$$\Rightarrow \psi_n(\xi) = N_n \varphi_n(\xi) \quad \lambda_n = 2n+1$$

NORMITUSVAAKIO $\Rightarrow E_n = (n + \frac{1}{2}) \hbar \omega$

* TIKAPOUOPERAATTORIT:

$$\left(\xi - \frac{d}{d\xi} \right) \varphi_n(\xi) = e^{-\xi^2/2} \left(\xi H_n(\xi) + \xi H_n(\xi) - H_n'(\xi) \right)$$

$$\stackrel{(2)}{=} e^{-\xi^2/2} (2\xi H_n(\xi) - 2n H_{n-1}(\xi)) = e^{-\xi^2/2} H_{n+1}(\xi)$$

$$\Rightarrow \varphi_{n+1}(\xi) = \left(\xi - \frac{d}{d\xi} \right) \varphi_n(\xi)$$

$$\begin{aligned}
 (\xi + \frac{d}{d\xi}) \varphi_n(\xi) &= e^{-\xi^2/2} (\cancel{\xi} \mu_n - \cancel{\xi} \mu_n' + \mu_n') = e^{-\xi^2/2} \mu_n' \\
 &= 2n e^{-\xi^2/2} H_{n-1}(\xi) = 2n \varphi_{n-1}(\xi) \\
 \Rightarrow \varphi_{n-1}(\xi) &= \frac{1}{2n} (\xi + \frac{d}{d\xi}) \varphi_n(\xi)
 \end{aligned}$$

$\varphi_{-1} \equiv 0$ PERUSTILA φ_0 SAADAAAN RATKAISEMALLA

$$(\xi + \frac{d}{d\xi}) \varphi_0(\xi) = 0$$

$$\Rightarrow \varphi_0 = N e^{-\xi^2/2}$$