Noncommutative QFT and Lamb shift

Mindaugas Lekaveckas

University of Helsinki, Department of Physical Sciences

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Layout

1. Classical theory of hydrogen atom
   - Dirac equation
   - Lamb shift in ordinary QED

2. Noncommutative QFT
   - General remarks about NC QFT
   - Lamb shift at tree level
   - Lamb shift from one loop corrections

Dirac equation

- Derived in 1928 by Dirac to describe relativistic motion of electron
- 4-component wavefunction $\psi(x)$
- Anticommuting 4x4 matrices were introduced

$$
\begin{aligned}
&\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1 & \\
&\{\alpha_i, \alpha_j\} = 0, & i \neq j \\
&\{\alpha_i, \beta\} = 0, &
\end{aligned}
\tag{1}
$$

- Dirac equation

$$
\left(i\hbar \frac{\partial}{\partial t} + i\hbar c \alpha_i \cdot \nabla_i - \beta m_e c^2 \right) \psi(x) = 0 \tag{2}
$$

- Generalization of Dirac equation to include interaction with $A_\mu(x)$

$$\begin{aligned} i\hbar\frac{\partial}{\partial t} = E \rightarrow E - e\phi \\ i\hbar c\vec{\alpha}\cdot\vec{\nabla} = c\vec{p} \rightarrow c\vec{p} - e\vec{A} \end{aligned} \tag{3}$$

- Generalized Dirac equation for fermionic particle is

$$[E - e\phi - \vec{\alpha}\cdot(c\vec{p} - e\vec{A}) - \beta mc^2]\psi(x) = 0 \tag{4}$$

- For hydrogen atom $\vec{A} = 0$ and $e\phi = V_{Coulumb}(r) = -e^2/r$

Solution of Dirac equation

- Energy levels turn out to be

$$E = m_e c^2 \left[1 + \frac{\alpha^2}{\left(\sqrt{\left(j+\frac{1}{2}\right)^2 - \alpha^2} + n' \right)^2} \right]^{-\frac{1}{2}} \tag{5}$$

- or expanding in powers of γ

$$E = m_e c^2 \left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4} \left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4} \right) + O(\alpha^6) \right] \tag{6}$$

$n = n' + j + \frac{1}{2}$ is principal quantum number, $\alpha \approx \frac{1}{137}$, j - total angular momentum quantum number

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- No dependence on l - there is no Lamb shift in Dirac theory!

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- In Dirac theory energy levels are degenerate with respect to quantum number l
- In 1947 transition $2S_{1/2} \rightarrow 2P_{1/2}$ (notation nl_j) in hydrogen atom was measured by Lamb and Retherford

Figure: Willis Lamb (1913-2008)

- Vertex modification and vacuum polarization are responsible for Lamb shift

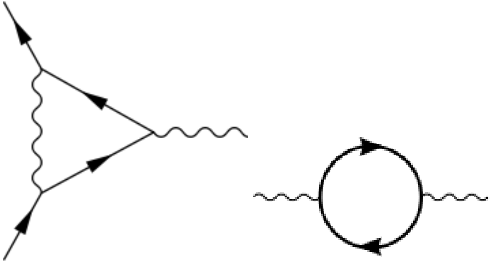

Figure: Vertex modification and vacuum polarization diagrams at one loop order

- Vertex function Γ_ρ changes due to first order loop effects

$$\Gamma_\rho = \gamma_\rho \to \gamma_\rho \left[1 + \frac{\alpha q^2}{3\pi m^2} \left(\ln \frac{m}{\mu} - \frac{3}{8} - \frac{1}{5} \right) \right] + \frac{i}{2m} \frac{\alpha}{2\pi} \sigma_{\rho\nu} q^\nu \tag{7}$$

and corresponding energy change

$$\delta E = \delta E^{(1)} + \delta E^{(2)}$$

$$= e\alpha \int d^3\vec{x} \psi^\dagger(\vec{x}) \left\{ \frac{1}{3\pi m^2} \left(\ln \frac{m}{\mu} - \frac{3}{8} - \frac{1}{5} \right) \Delta[A^0(\vec{x})] + \right.$$

$$\left. + \frac{i}{4\pi m} \vec{\gamma} \cdot \vec{E}(\vec{x}) \right\} \psi(\vec{x}) \tag{8}$$

$\delta E^{(1)}$

- when $l = 0$

$$\delta E^{(1)}_{n,l=0} = \frac{4m\alpha}{3\pi n^3}(Z\alpha)^4 \left[\left(\ln\frac{m}{2K} - \frac{3}{8} - \frac{1}{5} + \frac{5}{6}\right) + \ln\frac{K}{\langle E_n\rangle}\right] \quad (9)$$

- and when $l \neq 0$

$$\delta E^{(1)}_{n,l\neq 0} = \frac{4m\alpha}{3\pi n^3}(Z\alpha)^4 \ln\frac{Z^2\text{Ryd}}{\langle E_n\rangle} \quad (10)$$

- where Ryd = $m\alpha^2/2$ is hydrogen atom binding energy and $\ln\langle E_n\rangle$ is logarithmic energy average

- $\delta E^{(2)}$ contribution is

$$\delta E^{(2)}_{n,l,j} = \frac{\alpha(Z\alpha)^4}{2\pi n^3} m \frac{1}{2l+1} C_{j,l} \tag{11}$$

$$C_{j,l} = \begin{cases} 1/(l+1) & \text{if } j = l + 1/2 \text{ or } l = 0 \\ -1/l & \text{if } j = l - 1/2 \end{cases}$$

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- **so overall Lamb shift**

$$\boldsymbol{\delta E_{n,l,j} = \frac{4m\alpha}{3\pi n^3}(Z\alpha)^4 \begin{cases} \ln \frac{m}{2\langle E_{n,0}\rangle} + \frac{19}{30} & \text{for } l = 0 \\ \ln \frac{Z^2\text{Ryd}}{\langle E_{n,l}\rangle} + \frac{3}{8}\frac{C_{j,l}}{2l+1} & \text{for } l \neq 0 \end{cases}} \tag{12}$$

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- Lamb shift value for $2S_{1/2} \rightarrow 2P_{1/2}$ transition becomes

$$\Delta E_{Lamb} = \delta E_{2S_{1/2}} - \delta E_{2P_{1/2}} = 1052.1\text{MHz} \tag{13}$$

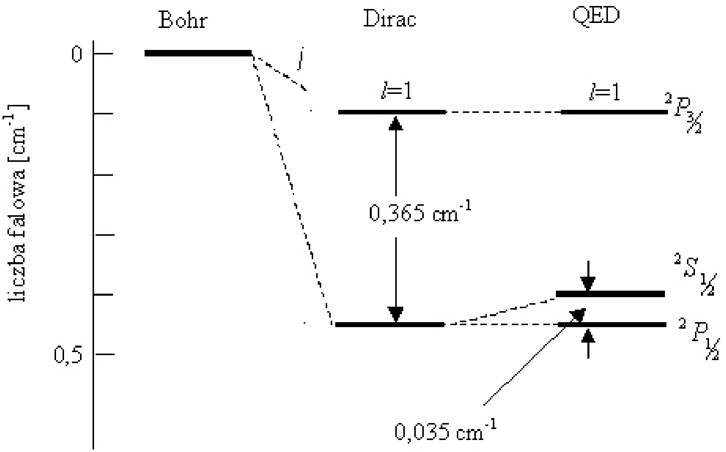

Figure: Energy differences in different theories, H. Haken, H.C.Wolf, Atomic Physics, 1996

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- At Planck Length $\lambda_P \approx 1.6 \times 10^{-35} m$ spacetime may be not uniform, so

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \qquad \mu, \nu = 0, .., 4 \tag{14}$$

- Three fundamental constants: $\hbar$ - from Quantum Mechanics, c - from Special Relativity, λ_P - from Noncommutative Geometry?
- hints from string theory, i.e., uncertainty relation

$$\Delta x = \frac{\hbar}{2}\left(\frac{1}{\Delta p} + l_s^2 \Delta p\right) \tag{15}$$

so we get $\Delta x_{min} = \hbar l_s^2$ which is consistent with (14).
- from principles of general relativity and Heisenberg uncertainty relation - large momentum required for measurement, which creates small black hole around point

Layout

- Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(\frac{p^2}{2m} + V(r)\right)\psi(x,t) \qquad (16)$$

- Coulomb potential for hydrogen atom

$$V(r) = -\frac{Ze^2}{r}, \qquad r \equiv \sqrt{x^2 + y^2 + z^2} \qquad (17)$$

- Commutators
  $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}$,
  $[\hat{p}_i, \hat{p}_j] = 0$,
  $[\hat{x}_i, \hat{x}_j] = i\theta_{ij}$,
  $[\hat{t}, \hat{x}_i] = 0$

- Idea: switch $\hat{x}_i \to x_i$ to “frame” where coordinates commute
- Realization:

$$x_i = \hat{x}_i + \frac{1}{2\hbar}\theta_{ij}\hat{p}_j \ , \qquad p_i = \hat{p}_i \tag{18}$$

- New coordinates

$$\begin{aligned} \left[x_i, x_j\right] &= 0 \\ \left[p_i, p_j\right] &= 0 \\ \left[x_i, p_j\right] &= i\hbar\delta_{ij} \end{aligned} \tag{19}$$

- Now Coulomb potential (17) becomes

$$\begin{aligned} V(r) &= -\frac{Ze^2}{\sqrt{(x_i - \theta_{ij}p_j/2\hbar)(x_i - \theta_{ik}p_k/2\hbar)}} \\ &= -\frac{Ze^2}{r} - Ze^2\frac{x_i\theta_{ij}p_j}{2\hbar r^3} + O(\theta^2) \\ &= -\frac{Ze^2}{r} - Ze^2\frac{\vec{L}\cdot\vec{\theta}}{4\hbar r^3} + O(\theta^2)\ , \end{aligned} \tag{20}$$

  where we denote $\theta_{ij} = \frac{1}{2}\epsilon_{ijk}\theta_k$ and use $\vec{L} = \vec{r}\times\vec{p}$.
- Perturbation expansion in θ:

$$\delta E^{NC} = -\langle nl'jj'_z|\frac{Ze^2}{4\hbar}\frac{L\cdot\theta}{r^3}|nljj_z\rangle \tag{21}$$

- Value for δE^{NC} can be found using usual quantum mechanics

$$\delta E^{NC}_{n,l,j_z} = -\frac{m_e c^2}{4}(Z\alpha)^4 \frac{\theta}{\lambda_e^2} j_z (1 \mp \frac{1}{2l+1}) \frac{1}{n^3 l(l+1/2)(l+1)} \delta_{ll'} \delta_{j_z j'_z} \tag{22}$$

with $j = l \pm 1/2$, and $\lambda_e \approx 2.43 \times 10^{-12}$ m

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- dependence not only on l but also on j_z
- Lamb shift $2S_{1/2} \to 2P_{1/2}$ is calculated to be

$$\Delta E^{NC}_{Lamb} = \delta E^{NC}_{n=2,l=0,j_z} - \delta E^{NC}_{n=2,l=1,j_z} = \frac{m_e c^2}{2^6}(Z\alpha)^4 \frac{\theta}{\lambda_e^2} j_z \quad (23)$$

- we can impose constraints on θ comparing to ordinary QFT Lamb shift result

$$\frac{\theta}{\lambda_e^2} \lesssim 10^{-7}\alpha \quad \text{or} \quad \theta \lesssim (10^4 \text{GeV})^{-2} \tag{24}$$

- not strong restriction, similar to constraints calculated for high energy experiments

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Vertex correction

- As in ordinary QED, two types of corrections: vacuum polarization and vertex correction
- Vertex function in NC QED is

$$\Gamma_\mu = e^{\frac{i}{2\hbar^2} p \times p\prime} \gamma_\mu = e^{-\frac{i}{2\hbar^2} p \cdot \tilde{q}} \gamma_\mu \ , \qquad (25)$$

where p and p' are the in-coming and out-going electron momenta, respectively, q_μ is the photon momentum: $p' - p = q$ and

$$p \times p' = p_\mu \theta^{\mu\nu} p'_\nu \ , \qquad \tilde{q}^\mu = \theta^{\mu\nu} q_\nu \ .$$

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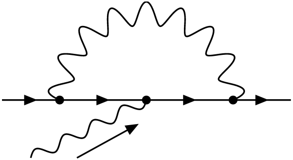
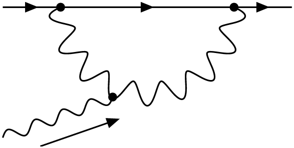
- At one loop level two corrections to Γ_μ:

Vertex corrections



(a) One loop QED-like correction to $\psi\bar{\psi}A_\mu$



(b) Nonabelian-type correction to $\psi\bar{\psi}A_\mu$

Vertex corrections

- Total Γ factor is

$$\Gamma^\mu = E_1\gamma^\mu + H_1(p'+p)^\mu + G_1\tilde{q}^\mu + E_2\gamma^\mu p\cdot\tilde{q} + H_3(p'+p)^\mu\gamma\cdot\tilde{q} \tag{26}$$

- Two contributions to potential $V(x) = -\langle P\rangle \cdot E(x)$ and $V(x) = -\langle\mu\rangle \cdot B(x)$ with

$$\langle P\rangle_i = 2iH_3\tilde{p}_i \ . \tag{27}$$

$$\langle\vec{\mu}\rangle = \frac{G_1}{i}\vec{\theta}\ , \qquad \theta_i \equiv \ \epsilon_{ijk}\theta_{jk}\ , \tag{28}$$

Vertex corrections

- G_1 and H_3 can be calculated

$$G_1 = \frac{\alpha e^{\frac{i}{2} p \times p'}}{\pi} \left(\frac{i m \gamma_{Euller}}{6} \right),$$

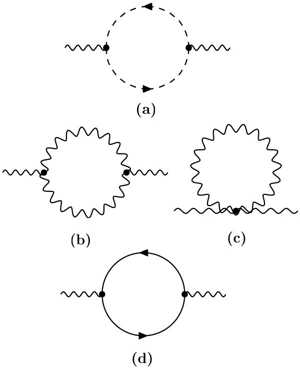
$$H_3 = \frac{-\alpha e^{\frac{i}{2} p \times p'}}{\pi} \left(\frac{3 i \gamma_{Euller}}{4} \right) . \quad (29)$$

- and total potential for vertex correction to first order in θ

$$V^{1\,Loop}_{\mathrm{NC\ vertex}} = -\frac{Ze^2}{4\pi} \gamma_{Euller} \alpha (3 - \frac{2}{3}) \frac{\vec{L}.\vec{\theta}}{\hbar r^3} . \quad (30)$$

Vacuum polarization

NC QED gauge field couples to itself as in ordinary non-Abelian theory, therefore to first order in θ vacuum polarization diagrams are



- Total potential correction including modification of propagator is

$$V^{1Loop}_{\text{prop.}}(r) = -\frac{Ze^2}{2\pi r}\frac{10\alpha}{3}\ln(\theta\Lambda^2) - Ze^2\frac{4\alpha}{15}\lambda_e^2\delta^3(r)\,, \qquad (31)$$

where Λ is momentum integration cut-off.

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- Summing vacuum polarization and vertex correction potentials we get

$$\Delta E^{NC}_{1Loop} = -\frac{1}{2\pi}m_e c^2(Z\alpha)^2\left[\frac{5\alpha}{3}\ln(\theta\Lambda^2)\frac{1}{n^2} + \right.$$

$$\left. + \frac{(Z\alpha)^2}{2}\frac{\theta}{\lambda_e^2}\gamma_E\alpha(3-\frac{2}{3})\frac{j_z(1\mp\frac{1}{2l+1})}{n^3 l(l+\frac{1}{2})(l+1)}\right]. \qquad (32)$$

- We notice that the second term of overall hydrogen atom energy shift from one loop corrections (32) is same as tree level result (22) but multiplied by additional factor $\gamma_{Euller}\alpha(3-\frac{2}{3})/\pi$. The first term is absent in tree level calculation and gives logarithmic dependence on momentum cut-off parameter Λ.

References

- M. Chaichian, M. M. Sheikh-Jabbari, A. Tureanu, *Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED*, *Phys. Rev. Lett.* **86** (2001) 2716, **hep-th/0010175**.
- M. Chaichian, M. M. Sheikh-Jabbari, A. Tureanu, *Comments on the Hydrogen Atom Spectrum in the Noncommutative Space, Eur.Phys.J.* **C36** (2004) 251-252, **hep-th/0212259**.
- I. F. Riad, M. M. Sheikh-Jabbari, *Noncommutative QED and Anomalous Dipole Moments, JHEP* **0008** (2000) 045.

Thank you for attention!