The realization of Very Special Relativity using noncommutative QFT

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Layout

Introduction

- Special relativity (SR) lies on assumption that laws of physics share many symmetries of Maxwell's equations.
- Most general group under which SR is invariant is 15-parameter conformal group $SU(2,4)$, but existence of massive particles and P, T violations restrict symmetries to Poincaré group.
- No violations of Lorentz symmetry observed so far.
- Neutrino mass – might be explained by Lorentz symmetry being non exact.

Poincaré group algebra

- Special Relativity – physics laws possess symmetry under Poincaré group transformations: $x'^{\mu} \to \Lambda^{\mu}_{\ \nu} x^{\nu} + a^{\mu}$.
- Infinitesimal transformation with $\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + \omega^{\mu}_{\ \nu}$ and $a^{\mu} = \epsilon^{\mu}$ is generated by $M^{\mu\nu}$ and P^{μ}, which form Lie algebra:

$$
\begin{aligned}
[P_{\mu}, P_{\nu}] &= 0 \\
[M_{\mu\nu}, P_{\alpha}] &= -i(\eta_{\mu\alpha} P_{\nu} - \eta_{\nu\alpha} P_{\mu}) \\
[M_{\mu\nu}, M_{\alpha\beta}] &= -i(\eta_{\mu\alpha} M_{\nu\beta} - \eta_{\mu\beta} M_{\nu\alpha} - \eta_{\nu\alpha} M_{\mu\beta} + \eta_{\nu\beta} M_{\mu\alpha}) \, .
\end{aligned}
$$

- $J^{i} = \epsilon^{ijk} M^{jk}$ are angular momentum operators and $K^{i} = M^{0i}$ – boost operators.

Classification of noncommutative QFTs

- Fundamental assumption

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu} \qquad \mu, \nu = 0, .., 4 \tag{1}$$

  where $\theta^{\mu\nu}$ is matrix with small components, likely of order of Planck length λ_P.
- Noncommutative theories can be classified according to parameter $\Lambda^4 \equiv \theta_{\mu\nu}\theta^{\mu\nu}$ which is related to the scale, where noncommutativity effects become important:
  i) $\Lambda^4 > 0$ – space-like (space-space) noncommutativity;
  ii) $\Lambda^4 < 0$ – time-like (time-space) noncommutativity;
  iii) $\Lambda^4 = 0$ – light-like noncommutativity.

Form of $\theta^{\mu\nu}$ can take three possible forms in order for noncommutative QFTs to be realizable:

1) *Constant* $\theta^{\mu\nu}$,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{2}$$

2) *Linear* $\theta^{\mu\nu}$,

$$[x^\mu, x^\nu] = iC^{\mu\nu}_\rho x^\rho , \tag{3}$$

3) *Quadratic* $\theta^{\mu\nu}$,

$$[x^\mu, x^\nu] = \frac{1}{q} R^{\mu\nu}_{\rho\sigma} x^\rho x^\sigma , \tag{4}$$

where $\theta^{\mu\nu}$, $C^{\mu\nu}_\rho$ and $R^{\mu\nu}_{\rho\sigma}$ are constant antisymmetric 'matrices'.

Layout

- Very Special Relativity (VSR) is defined as symmetry under certain subgroups of Poincaré group, containing space-time translations and a proper subgroup of Lorentz group $SO(1,3)$ with the property that when supplemented with parity, CP or time-reversal T it enlarges to the full Lorentz group.
- Space translation invariance preserved – generators of translations P_μ are included $\Rightarrow$ momentum $\vec{p}$ can be used as state label.
- For minimal version of VSR, subgroup $T(2)$ of the Lorentz group is also included.
- $T(2)$ is generated by

$$T_1 = K_x + J_y \qquad \text{and} \qquad T_2 = K_y + J_x \ , \tag{5}$$

where J_i are generators of rotations and K_i are generators of boosts with $i = x, y, z$.

Invariant tensors

$T(2)$ can be identified with the translation group on two dimensional plane and admits many invariant tensors, i.e.,

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{6}$$

F may be thought of as the field-strength for a zero frequency electromagnetic wave with linear polarization in the x-direction.

The other larger versions of VSR are obtained by adding one or two Lorentz generators to $T(2)$, which have geometric realizations on two dimensional plane:

1. $E(2)$, the 3-parametric group of two dimensional Euclidean motion, generated by $T_1,\ T_2$ and J_z, with the structure:

$$[T_1, T_2] = 0, \quad [J_z, T_1] = iT_2, \quad [J_z, T_2] = -iT_1;$$

2. $HOM(2)$, the group of orientation-preserving similarity transformations, or homotheties, generated by $T_1,\ T_2$ and K_z, with the structure

$$[T_1, T_2] = 0, \quad [T_1, K_z] = iT_1, \quad [T_2, K_z] = iT_2;$$

3. $SIM(2)$, the group isomorphic to the four-parametric similitude group, generated by $T_1,\ T_2,\ J_z$ and K_z, with the structure

$$[T_1, T_2] = 0, \quad [J_z, T_1] = iT_2, \quad [J_z, T_2] = -iT_1,$$

$$[T_1, K_z] = iT_1, \quad [T_2, K_z] = iT_2, \quad [J_z, K_z] = 0.$$

Layout

Problem of representation content

- Problem of VSR group representation content – group $T(2)$ is Abelian, so admits only one-dimensional representations.
- Representations of Lorentz group are automatically representations of VSR, but reciprocal statement is not correct.
- When we enhance VSR with generators of P, T or CP, we do not get representations of full Lorentz group.

Twisted Poincaré symmetry

- Solution – instead of working with Lie algebra structure, let us introduce (deformed) Hopf algebras.
- Some transformations of Hopf algebras leave commutation relations untouched (that of Lie algebra), but affect other properties of Hopf algebras, i.e, co-algebra structure.
- Hopf algebras are so called twist deformed algebras, which generate twisted Poincaré transformations.
- It was shown that theories on noncommutative space-time are invariant under *twisted Poincaré transformations* which are noncommutative analogs of Poincaré transformations.

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- Hopf algebras are so called twist deformed algebras, which generate twisted Poincaré transformations.
- It was shown that theories on noncommutative space-time are invariant under *twisted Poincaré transformations* which are noncommutative analogs of Poincaré transformations.
- Due to *twisted Poincaré transformations* we can still use usual Lorentz group representations and label particles with mass m and spin s.

Realization of VSR

- It turns out that minimal VSR (with $T(2)$ Lorentz subgroup) can be realized with constant $\theta^{\mu\nu}$, $E(2)$ with linear $\theta^{\mu\nu}$and $SIM(2)$ with quadratic $\theta^{\mu\nu}$.
- We focus on minimal VSR and set about determining the structure of $\theta_{\mu\nu}$ which is covariant under $T(2)$ transformations.

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- Elements of $T(2)$:

$$\Lambda_1 = e^{i\alpha T_1} \quad \text{and} \quad \Lambda_2 = e^{i\beta T_2} \,, \tag{7}$$

- covariance of $\theta_{\mu\nu}$ under Λ_1 and Λ_2 requires that

$$\Lambda_i{}^{\mu}{}_{\alpha} \Lambda_i{}^{\nu}{}_{\beta} \theta^{\alpha\beta} = \theta^{\mu\nu}, \quad i = 1, 2 \,, \tag{8}$$

- or infinitesimally,

$$T_i{}^{\mu}{}_{\alpha} \theta^{\alpha\nu} + T_i{}^{\nu}{}_{\beta} \theta^{\mu\beta} = 0, \quad i = 1, 2. \tag{9}$$

The matrix realization of the generators T_1 and T_2 can be easily worked out using generators for K_x, J_y, K_y and J_x:

$$T_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \tag{10}$$

and

$$T_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}. \tag{11}$$

Form of $\theta^{\mu\nu}$

- We can find solution for $\theta_{\mu\nu}$:

$$\begin{aligned} \theta^{0i} &= -\theta^{3i}, \quad i = 1, 2, \\ \theta^{\mu\nu} &= 0, \quad \text{for all other components.} \end{aligned} \tag{12}$$

- Result holds for any x-dependence of $\theta^{\mu\nu}$.
- $\Lambda^4 = \theta^{\mu\nu}\theta_{\mu\nu} = 0$, so for light-like noncommutativity, $\theta^{\mu\nu}$ is covariant under $T(2)$.

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- Result holds for any x-dependence of $\theta^{\mu\nu}$.
- $\Lambda^4 = \theta^{\mu\nu}\theta_{\mu\nu} = 0$, so for light-like noncommutativity, $\theta^{\mu\nu}$ is covariant under $T(2)$.
- VSR can be realized on light-like noncommuting space-time!

Layout

- In last presentation (24.02.2009) Lamb shift calculation for space-like noncommutativity ($\theta_{0i} = 0, \Lambda^4 = \theta^{\mu\nu}\theta_{\mu\nu} > 0$) was reviewed.
- We will calculate now Lamb shift for light-like ($\Lambda^4 = \theta^{\mu\nu}\theta_{\mu\nu} = 0$) noncommutativity.

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- We will calculate now Lamb shift for light-like ($\Lambda^4 = \theta^{\mu\nu}\theta_{\mu\nu} = 0$) noncommutativity.
- Again we aim for transformations of $\hat{x}_\mu \to x_\mu$ and $\hat{p}_\mu \to p_\mu$ in first order of $\theta^{\mu\nu}$ that would give:
  1. $[x_\mu, x_\nu] = 0$,
  2. $[x_i, p_j] = i\hbar\delta_{ij}$,
  3. $[p_i, p_j] = 0$,
  4. $[x_0, p_i] = 0$.

  and we could treat $\theta^{\mu\nu}$ parameter as perturbation.

- Assume general transformations are of the form

$$x_\mu = \hat{x}_\mu + \alpha_{\mu\nu}\hat{p}^\nu \ , \qquad p_\mu = \hat{p}_\mu \ . \tag{13}$$

- Evaluating

$$\left[x_i, p_j\right] = i\hbar\delta_{ij} + \alpha_{i0}\left[\hat{H}, \hat{p}_j\right] \tag{14}$$

  gives $\boldsymbol{\alpha_{i0} = 0}$.

- From

$$\left[x_0, p_j\right] = \alpha_{00}\left[\hat{H}, \hat{p}_j\right] \tag{15}$$

  we get that $\boldsymbol{\alpha_{00} = 0}$.

- α_{ij} and α_{0i} turn out to be nonzero and have values:

$$\boldsymbol{\alpha_{ij} = \frac{1}{2\hbar}\theta_{ij}, \qquad \alpha_{0i} = \frac{1}{\hbar}\theta_{0i}.} \tag{16}$$

So transformation rules are

- for space coordinate operators

$$x_i = \hat{x}_i + \frac{1}{2\hbar}\theta_{ij}\hat{p}_j \; ; \tag{17}$$

- for time operator

$$t = \hat{t} + \frac{1}{\hbar}\theta_{0i}\hat{p}_i \; . \tag{18}$$

Since transformation rules for x_i are the same as for space-like noncommutativity and potential does not include time explicitly, we conclude that <u>Lamb shift is the same in light-like and space-like noncommutative QFTs</u>!

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Since any realistic potential does not include time explicitly $\Rightarrow$ space-like and light-like noncommutative theories have analogous perturbative predictions up to $O(\theta)$.

References

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