MOKA[1] - Many Body Physics

K. Enqvist and K. Kajantie

December 5, 2007

[1]From Finnish for Many Body Physics: Monen Kappaleen fysiikka

Chapter 7

Landau theory

7.1 Formulation of the Landau theory

Spin models teach us that in many cases one can describe the various phases of the system by an order parameter (or a set of order parameters), which gauges the properties of the system in some average sense. Hence we have an effective description: one can imagine a more complete theory containing a large number of degrees of freedom ϕ, part of which are integrated over under the condition that one of their combinations $M(\phi)$ be kept fixed:

$$e^{-\beta F(T,V,M)} = \int \mathcal{D}\phi \, e^{-\beta H(\phi)} \delta(M - M(\phi)) \ . \tag{7.1}$$

If one further integrates over M one obtains the partition function of the full theory:

$$Z = e^{-\beta F(T,V)} = \int \mathcal{D}\phi \, e^{-\beta H} = \int \mathcal{D}M[\mathbf{x}] \, e^{-\beta F(T,V,M[\mathbf{x}])}. \tag{7.2}$$

Effective approaches are guided by simplicity and symmetry principles and best justified by their workings. A formal *derivation* of e.g. $F(T,V,M[\mathbf{x}])$ in Eq. (7.2) is often very difficult if not impossible (cf. BCS-theory $\Rightarrow$ Ginzburg-Landau-theory). Therefore we have to be satisfied with reasonable, physically motivated assumptions. To this end, let us write the free energy of a statistical system as a functional of some order parameter $M(\boldsymbol{x})$ (assumed here to be real for simplicity):

$$\begin{aligned} F[M(\boldsymbol{x})] &= \int d^3x [\frac{K}{2} |\boldsymbol{\nabla} M|^2 + \frac{1}{2} a M^2(\boldsymbol{x}) + \frac{1}{4} b M^4(\boldsymbol{x}) - hM(\mathbf{x}) + cM^6(\mathbf{x}) + \ldots] \quad (7.3) \\ &\equiv \int d^3x \big[\frac{K}{2} |\boldsymbol{\nabla} M|^2 + V(M(\mathbf{x}))\big], \quad (7.4) \end{aligned}$$

where the parameters satisfy

$$K > 0, \qquad a, \lesseqgtr 0 \qquad b > 0 \text{ (if } c = 0), \qquad h \gtrless 0, \qquad c > 0 \ldots,$$

Eq. (7.4) defines the Landau model. We have used here a condensed matter notation: in particle physics one would write $K = 1$ (the "canonical normalisation of the kinetic term", which can always be done in case of a constant K by the field redefinition $M \to M/\sqrt{K}$), $a = m^2$, $b = \lambda$, etc. The parameters may in general depend on T.

We could as well think of Eq. (7.4) as defining a 3d field theory with the action F. The functional integral

$$Z[J] = e^{-F(J)} = \int \mathcal{D}M[\mathbf{x}]\, e^{-F(M[\mathbf{x}])+\int d^3x J(\mathbf{x})M(\mathbf{x})} \tag{7.5}$$

would then give all the Green's functions or operator expectation values, i.e., all the physics content, of this theory. In the original Landau theory the argument for the validity of (7.4) was that near the phase transition M is small and higher powers can be neglected. In effective field theories it often happens that, even for dimensional reasons, higher powers come together with inverse powers of some mass scale and can thus be neglected when this mass scale is large.

If $h = 0$, the only symmetry of the action Eq. (7.4) or the theory Eq. (7.2) is $M \to -M$ (the group Z(2)). This is not accidental but rather reflects the fact that we have not included odd terms like M^3 in the free energy. Thus imposing a symmetry limits the possible form of the free energy in an essential manner. Instead on Z(2), we could impose a global U(1) symmetry, in which case the order parameter has to be complex $M = M_1 + iM_2$. If we gauge the U(1) symmetry, i.e. demand that it is local and valid at each point $\mathbf{x}$ separately, we have to introduce a gauge field so that effectively we arrive at a theory describing the interaction of an order parameter with a massless gauge field (photon); this is the Landau-Ginzburg theory of superconductivity. We could also have an order parameter which is a vector or a tensor.

Apart from the change in notation, the theory defined here is, of course, nothing but the scalar field theory discussed earlier in Section 3.

Scaling to dimensionless variables. Clearly F/T is dimensionless, but

$$\frac{1}{T}F[M(\boldsymbol{x})] = \int d^3x \left(\frac{K}{2T}|\boldsymbol{\nabla} M|^2 + \frac{1}{2}\frac{a}{T}M^2 + \frac{1}{4}M^4\right) \tag{7.6}$$

contains quantities which are known but vary from problem to problem ($K = \frac{\hbar^2}{2m}$ or 1, M = magnetisation or wave function, ...). It is very useful to scale out dimensions. From dim$(F/T) \equiv [F/T]$=1 one finds that the dimensionalities are

$$[a] = [\frac{K}{x^2}], \quad [M^2] = [\frac{T}{Kx}], \quad [b] = [\frac{K^2}{Tx}] \quad \text{(this depends of } d = 3!\text{)}.$$

Using these units we have the simple dimensionless form

$$\int d^3x \left(\frac{1}{2}|\boldsymbol{\nabla} M|^2 + \frac{1}{2}aM^2 + \frac{1}{4}bM^4\right). \tag{7.7}$$

7.2 Z(2) symmetry: the minimum energy configurations

We now have the theory defined by the path integral

$$Z = e^{-\beta F} = \int \mathcal{D}M(\mathbf{x})\, e^{-\beta F(M(\mathbf{x}))}. \tag{7.8}$$

To minimize F one should minimize the whole integral. In Landau theory one usually assumes that it is a good approximation to minimize the integrand, i.e., find the extrema of $F(M(\mathbf{x}))$. This approximation has many names: saddle point, tree level, mean field, classical field approximation, as already discussed in some detail in Chapter 1. So let us simply take the first functional derivative of F with respect to $M(\mathbf{x})$ (do first a very common partial integration based on $(f')^2 = -f \cdot f'' + d(f \cdot f')/dx$ and assume that the boundary term vanishes). Thus, as before, we arrive at the equation of motion

$$\begin{aligned} \frac{\delta F[M(\boldsymbol{x})]}{\delta M(\boldsymbol{y})} &= \frac{\delta}{\delta M(\boldsymbol{y})} \int d^3x \Big[\frac{K}{2} M(\boldsymbol{x})(-\boldsymbol{\nabla}^2) M(\boldsymbol{x}) + V(M(\boldsymbol{x}))\Big] \\ &= -K\boldsymbol{\nabla}^2 M(\boldsymbol{y}) + V'(M(\boldsymbol{y})) = 0 \end{aligned} \tag{7.9}$$

Equally, one may use the equation of motion derived from the Lagrangian $L(M, \partial_i M)$:

$$\frac{\partial L}{\partial M} - \partial_i \frac{\partial L}{\partial \partial_i M} = V'(M) - \partial_i[K\partial_i M]) = 0. \tag{7.10}$$

To find the physical configuration $M(\mathbf{x})$, we can either *numerically* minimize

$$\int d^3x [\frac{K}{2} (\boldsymbol{\nabla} M)^2 + V(M)] \tag{7.11}$$

or solve, most likely also numerically, the equation of motion

$$K\nabla^2 M = V'(M).$$

Remark. In searching for the classical solution one may also start from the "Langevin-type" kinetic equation discussed in connection with numerical methods and neglect the stochastic term there, i.e., apply the equation

$$\frac{dM(\boldsymbol{x},t)}{dt} = -\Gamma \frac{\delta F(M(\boldsymbol{x},t))}{\delta M(\boldsymbol{x},t)} \tag{7.12}$$

where $\Gamma > 0$ is some constant and t is a fictitious computer time. This just expresses the natural fact that as long one is not in equilibrium, $\delta F/\delta M \neq 0$, the time derivative of M is proportional to the deviation from equilibrium. The role of the stochastic term was to thermalise the system and to keep it from falling into the classical solution.

To solve the equation of motion (7.2) one needs the boundary conditions (BC). These define the physical system one is considering. We shall go through a number of alternatives.

- **Simplest case, no BC forces $M(\boldsymbol{x})$ to vary**

  In this case any variation of M increases the action and F clearly is minimised by $\nabla^2 M = 0$. $M(\boldsymbol{x})$ = constant at least satisfies this. Then the equation of motion reads

$$V'(M) = aM + bM^3 - h = 0. \tag{7.13}$$

Writing here (phenomenologically) $a = \alpha(T - T_c)$ (Landau's notation) and taking in turn $a = 0$, $b = 0$ and $h = 0$ we can motivate the existence and evaluate the mean field values of three relevant critical indices.

(i) Take first $h = 0$ and plot the potential for $a > 0$ and $a < 0$ as in Fig. 7.1.

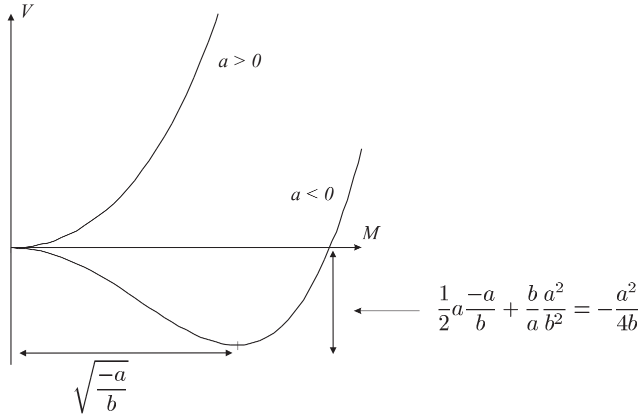

Figure 7.1: The potential for positive and negative a.

This describes phenomenologically spontaneous symmetry breaking (SSB), a cornerstone of modern theoretical physics. If $\langle M\rangle$ is the value of M at the minimum, the expectation value of M, one has

$a > 0:$	$\langle M\rangle = 0$	symmetric under $M \leftrightarrow -M$
$a < 0:$	$\langle M\rangle = \sqrt{\frac{-a}{b}}$	symmetry$M \leftrightarrow -M$ broken

In the symmetry broken phase $a = \alpha(T - T_c)$ and

$$T < T_c: \quad \langle M\rangle = \sqrt{\frac{\alpha}{b}}\sqrt{T_c - T} \sim t^{\frac{1}{2}} \tag{7.14}$$

which expresses the fact that the mean field value of the critical index β, $\langle M\rangle \sim |T_c - T|^\beta$ near T_c is 1/2. Pictorially:

(ii) Next take $b = 0$ so that the potential is $V = \frac{1}{2}aM^2 - hM$ and the condition for the minimum becomes $aM - h = 0$. Now the Z_2 symmetry $M \to -M$ is broken, h is a symmetry breaking parameter. Even the smallest non-zero h is enough to break the

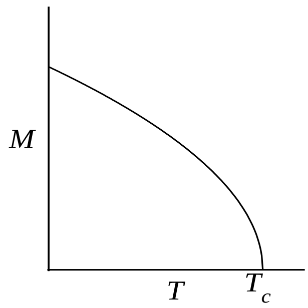

Figure 7.2: $M \sim \sqrt{T_c - T}$. Note this is valid only near $M \approx 0$, near T_c.

symmetry. There are many instances of this phenomenon, e.g., in QCD the smallest non-zero quark masses break the chiral symmetry.

For $a > 0$ the potential looks as shown in Fig.7.3 and the value at the minimum becomes

$$\langle M \rangle = \frac{h}{a} \approx \frac{h}{T - T_c}$$

so that the susceptibility, i.e., the response of the system to a small magnetic field becomes

$$\chi_M = \frac{\partial M}{\partial H} \sim \frac{1}{T - T_c} \sim t^{-\gamma}, \quad \gamma = 1. \tag{7.15}$$

Again we have derived the mean field value of one critical index, that corresponding to magnetic susceptibility, $\gamma = 1$.

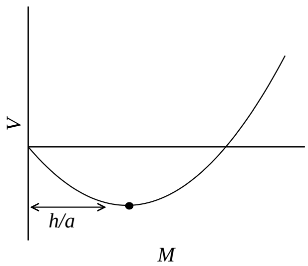

Figure 7.3: The potential near $M = 0$ for $b = 0, a > 0$.

(iii) Finally, take $a = 0$. Then the condition for the minimum is $h = bM^3$ so that

$$\langle M \rangle \sim h^{1/3} \sim h^{1/\delta}$$

where we have defined the third critical index δ and found its mean field value $\delta = 3$.

- **The case with boundary conditions: interface**.

  Take now $h = 0$ so that

$$\langle M \rangle = \pm\sqrt{\frac{-a}{b}} \equiv M_0 \tag{7.16}$$

and assume that one of the directions is special so that for $x \to +\infty$ the system is in the minimum $\langle M\rangle = +\sqrt{-a/b}$ and for $x \to -\infty$ in $\langle M\rangle = -\sqrt{-a/b}$. These two minima are shown in Fig.7.2 and we can set the BC so that the system moves from one minimum to another, i.e., $M(x, y, z) \equiv M(x)$ to varies as a function of x, Fig. 7.2.

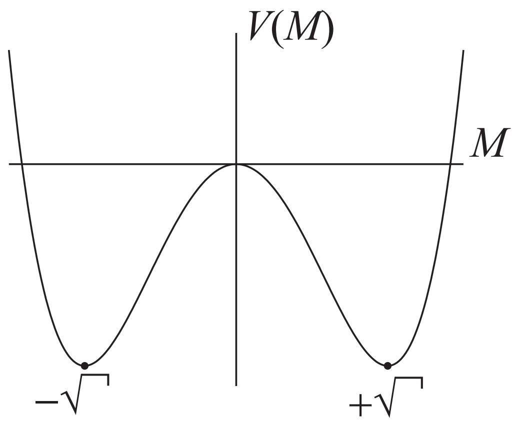

Figure 7.4: Double well

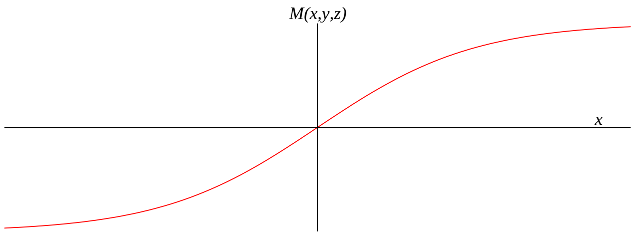

Figure 7.5: Variation $\sim \tanh(x)$.

It is easy to solve the equation of motion $-K\nabla^2 M + aM + bM^3 = 0, \quad a < 0$, analytically. Introducing the dimensionless variables

$$\begin{cases} M = \sqrt{\frac{-a}{b}}\hat{M} \\ x = \sqrt{\frac{K}{-a}}\hat{x} \end{cases}$$

the equation of motion becomes

$$\hat{M}''(\hat{x}) + \hat{M}(\hat{x}) - \hat{M}^3(\hat{x}) = 0. \tag{7.17}$$

This is solved by

$$\hat{M}(\hat{x}) = \tanh\frac{\hat{x}}{\sqrt{2}} \equiv \hat{M}_{\mathrm{extr}}(\hat{x}). \tag{7.18}$$

Physically, what we have obtained is an interface or a domain wall.

Evaluating the free energy (7.4) for this particular form of M gives the free energy of the interface ($F = -\rho V + \sigma A$), often called the interface tension σ:

$$\begin{aligned} F[M_{extr}(\boldsymbol{x})] &= \int_{-\infty}^{\infty} dx \, \overbrace{dy\, dz}^{area} \, [\frac{K}{2}(M'_{\rm extr}(\boldsymbol{x}))^2 + V(M_{\rm extr})] \\ &= \text{Area} \cdot \frac{2\sqrt{2}}{3} \frac{(-a)^{3/2}}{b} \equiv A \cdot \sigma. \end{aligned} \tag{7.19}$$

This is the "classic", "saddle point", "tree level", "mean field", extremal solution. The effect of quantum fluctuations around it has been computed for small b (Münster, Nucl. Phys. B324(1989)630):

$$\sigma \to \sigma[1 - \frac{b}{16\pi^2}(\frac{3}{4} + \frac{\pi\sqrt{3}}{2} + \mathcal{O}(b^2)]. \tag{7.20}$$

Both quantum and thermal fluctuations can be important near the critical temperature. To demonstrate this, let us ask what is the probability for the system to fluctuate back to $M = 0$ from the minimum energy state $M = M_0$ in Eq. (7.16) at $T < T_c$. Since the probability is proportional to the Boltzmann factor, and for a constant order parameter the free energy is density $\times$ volume so that we may write

$$P(M = 0; T < T_c) \sim e^{-\beta V[f(M_0) - f(0)]} \; . \tag{7.21}$$

Hence the fluctuation probability is vanishingly small with $P(M = 0; T < T_c) \ll 1$ when the exponent is smaller than 1. To be concrete, let us take $a = \mu_0^2(1 - T/T_c)$ whence we find that fluctuations back to the unbroken phase end when

$$\begin{aligned} & \beta V \frac{\mu_0^4}{4b} \left(1 - \frac{T}{T_c}\right)^2 \gg 1 \\ \to \quad & T_c - T \gg \frac{16 b^2 T_c^3}{K^3 \mu_0^2} \equiv \Delta T_c \; , \end{aligned} \tag{7.22}$$

where we have taken the volume factor V to be given by the correlation volume ξ^3, where $\xi = \sqrt{-K/a}$ as this is the spatial scale where the order parameter solution connecting the two phases changes by a relative amount of $\mathcal{O}(1)$. Thus very close to the critical temperature, in a region ΔT_c, the behaviour of a system is typically difficult to compute because the fluctuations are large.

Order parameter coupled to gravity. In the normal laboratory environment gravity plays no role. However, at very large scales gravity is important and one has to consider phase transitions in the presence of gravity. Perhaps the most striking example can be found in cosmic inflation, which is a period of superluminal expansion in the very early universe. It is driven by a coherent scalar field $\phi = \phi(t)$, the inflaton, which can be viewed as an order parameter of some early phase transition. If the potential is flat enough, as depicted in Fig. XXXXXX, the inflaton rolls slowly from $\phi \approx 0$ towards the minimum of the potential at $\phi = \phi_0$. Slow rolling means that the kinetic energy is much less than the potential energy, $\dot{\phi}^2 \ll V(\phi) \approx V_0$, which will hold until ϕ has grown to some value ϕ_* at $t = t_*$, after which the inflaton field starts to oscillate. During the slow roll the energy and the pressure are given by

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\phi}^2 + V \approx V \\ p &= \frac{1}{2}\dot{\phi}^2 - V \approx -V \end{aligned} \tag{7.23}$$

so that the equation of state is $\rho \approx -p$; effectively, there is a large cosmological constant. Then, assuming a flat universe with a Friedmann-Robertson-Walker metric $ds^2 = dt^2 - R^2(t)d\mathbf{x}^2$ the Einstein equation reads

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho \approx \text{const.} \equiv H_0^2 \ , \tag{7.24}$$

the solution to which is

$$R(t) = R_0 e^{H_0 t} \ . \tag{7.25}$$

Thus the "size" of the universe becomes exponentionally large with the number of e-folds of the scale factor R given by $N = H_0 t_*$. For an adiabatic expansion $RT =$ const. so that during inflation the temperature of the universe $T \to 0$. After inflation ends at $t \approx t_*$, the inflation energy is assumed to be dissipated by decay, and as a consequence the universe reheats.

Inflation was initially introduced (in 1981) because it solves some of the naturalness problems of the Big Bang theory. However, its most important feature is that it can expalain the origin of structure in the universe. During inflation ϕ is subject to quantum fluctuations $\delta\phi$, which give rise to a calculable spectrum of density perturbations since (schematically) $\delta\rho = V'\delta\phi$. The spectrum can be measured from the temperature fluctuatations of the cosmic microwave background since $\rho_\gamma \propto T^4$ implies that $4\delta T/T = \delta\rho_\gamma/\rho_\gamma$. The temperature fluctuatations have been measured very accurately e.g. the WMAP satellite, and the results agree well with the inflationary predictions.

7.3 U(1) symmetry: complex order parameter

Let us now consider a complex order parameter with $M = \frac{1}{\sqrt{2}}(M_1 + iM_2)$. We write the free energy (or the action) in a form which is invariant under rotations in the complex plane. Instead of Z(2) the symmetry then is U(1), defined by

$$M \to e^{i\chi} M. \tag{7.26}$$

As long as the parameter χ is constant, this is called a *global* U(1) symmetry. In Sect. 8.5 we shall assume χ to be some arbitrary function $\chi(\mathbf{x})$ which leads to local or gauged U(1).

In case of the global U(1) symmetry, the free energy of the Landau model reads

$$F = \int d^3x \Big[K \boldsymbol{\nabla} M^* \cdot \boldsymbol{\nabla} M + a|M|^2 + b|M|^4 \Big] \ . \tag{7.27}$$

Note that the factors $\frac{1}{2}, \frac{1}{4}$ are built in the definition of M in terms of real variables. The parameter K could be absorbed into the definition of M by rescaling $M \to M/\sqrt{K}$. From Eq. (7.27) one derives the EOM

$$\frac{\delta F}{\delta M^*} = 0 \quad \Rightarrow -K\nabla^2 M + aM + 2b|M|^2 M = 0 \ . \tag{7.28}$$

For $a < 0$ this has a particularly interesting special solution: a vortex line or a string. For the global U(1) symmetry these are called global vortices. These are cylindrical configurations. Hence it is natural to use the cylindrical coordinates $\mathbf{x} \to r, \phi, z$, see Fig. 7.6.

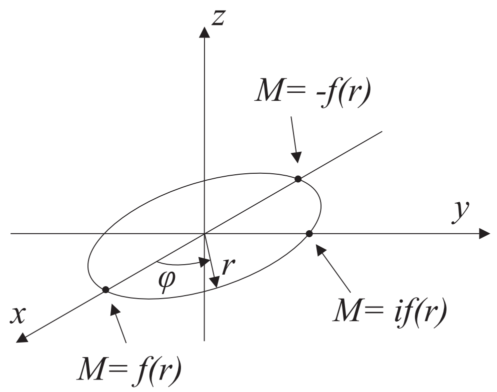

Figure 7.6: Coordinates for a global string

In order to find the string solution let us adopt the Ansatz

$$M(\boldsymbol{x}) = \frac{1}{\sqrt{2}} f(r) e^{in\phi} \quad n = \text{ integer} \tag{7.29}$$

with the boundary conditions

$$f(0) = 0, \quad f(\infty) = \sqrt{\frac{-a}{b}}. \tag{7.30}$$

Due to $f(0) = 0$ one needs not to worry about ϕ being undefined there. Using

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \tag{7.31}$$

the EOM becomes

$$-K\frac{1}{r}\frac{d}{dr}(r\frac{dM(r)}{dr}) + \frac{Kn^2}{r^2}M(r) + aM(r) + 2bM^3(r) = 0 \ . \tag{7.32}$$

Introducing again dimensionless variables by $M = \sqrt{-a/b}\hat{M}$ and $r = \sqrt{K/(-a)}\hat{r}$ yields

$$-K\frac{-a}{K}\frac{1}{r}\frac{d}{dr}(r\hat{M})\underline{\sqrt{\frac{-a}{b}}} + Kn^2\frac{-a}{K}\underline{\sqrt{\frac{-a}{b}}}\frac{\hat{M}}{\hat{r}^2} + \underline{\sqrt{\frac{-a}{b}}}a\hat{M} + b\frac{-a}{b}\underline{\sqrt{\frac{-a}{b}}}\hat{M}^3 = 0. \tag{7.33}$$

Cancelling some terms and omitting now hats from the dimensionless variables gives

$$-\frac{1}{r}\frac{d}{dr}(rM') + n^2\frac{M}{r^2} - M + M^3 = 0 \tag{7.34}$$

or

$$\frac{1}{r}\frac{d}{dr}(rM'(r)) + (1 - \frac{n^2}{r^2})M - M^3 = 0 \tag{7.35}$$

and finally

$$M'' + \frac{1}{r}M' - \frac{n^2}{r^2}M + M - M^3 = 0 \ . \tag{7.36}$$

Comparing with Eq. (7.17) one notices that two new terms have appeared (the 2nd and 3rd terms). These imply that $\tanh(r/\sqrt{2})$ is not a solution. One has to find the solution numerically, and the outcome is an $M(r)$ growing monotonically from 0 at $r = 0$ to 1 at $r = \infty$, somewhat resembling the $r > 0$ half of $\tanh(r)$ (see, e.g., Fig. 55.1 of Fetter-Walecka).

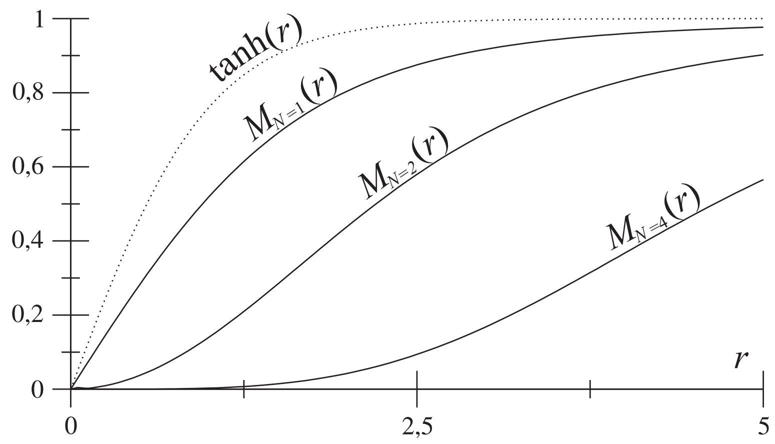

Figure 7.7: Radial dependences of a global string.

Now the free energy of the extremal configuration is proportional to the string length; it is the string tension. It has to be computed numerically, but it is even more important to estimate it parametrically:

$$\begin{aligned} F_{\text{extr}} &= \int dz \int dx\, dy\, \left[K|\boldsymbol{\nabla} M_{\text{extr}}|^2 + V(M_{\text{extr}})\right] \\ &= \text{Length} \cdot \text{Area} \cdot \text{free energy density} \cdot \text{number}. \end{aligned} \tag{7.37}$$

For Area we have the natural estimate Area $= K/(-a)$ (see the definitions of $\hat{r}$ above, one can also say Area = correlation length^2). Similarly free energy density = the value of $V(M)$ at the broken minimum $= -a^2/(4b)$. Thus we estimate

$$\frac{F}{\text{Length}} \equiv \text{Tension} = \text{number} \cdot K \cdot \frac{-a}{b} = \underbrace{\text{number}}_{\mathcal{O}(1)} \cdot K \cdot \langle M \rangle^2. \tag{7.38}$$

Example: Rotation of a bucket of superfluid. Because of superfluidity, the system cannot rotate as a whole, but if it is forced to rotate, it carries the angular momentum along global vortex lines, distributed in a hexagonal lattice (compare Abrikosov vortices in type II superconductors and magnetic fields) in the center of which the matter is in the normal symmetric state. If the superfluid wave function is $\psi = f(r)e^{i\theta}$, the superfluid density n_s and velocity of the matter $\mathbf{v}_s$ are given by

$$n_s = f^2(r), \qquad \mathbf{v}_s = \frac{\hbar}{m}\nabla\theta.$$

Note that this implies $\nabla\times\mathbf{v}_s=0$ everywhere except at $r=0$ (Feynman 1955). If simply $\theta=\phi$, the azimuthal angle, one has

$$|\mathbf{v}_s|=\frac{\hbar}{mr},\qquad \frac{h}{m_{\mathrm{He}}}=1.0\cdot 10^{-7}\frac{\mathrm{m}^2}{\mathrm{s}}.$$

Thus the velocity decreases with r, in contrast to rigid rotation. More generally, $\theta=n\phi$, $n=$ integer, but only $n=1$ is stable. For the tension of a superfluid vortex one obtains

$$\frac{E}{\Delta z}=\int d^2x\frac{1}{2}mn_sv_s^2\approx\frac{\pi\hbar^2n_s}{m}\log\frac{L}{\xi},$$

where L is the size of the system or the distance between vortices and ξ is the correlation length, width of the vortex. For He_4 the tension is some 10^{-12} N, in a neutron star the tension can be 10^4 N. One also has, for one vortex,

$$\oint\mathbf{v}_s\cdot d\mathbf{l}=\int\nabla\times\mathbf{v}_s\cdot d\mathbf{A}=\frac{h}{m}.$$

7.4 Symmetry currents

Imposing a *symmetry* on the theory means that the Lagrangian $L(\phi_k,\partial_i\phi_k)$ is invariant under the infinitesimal symmetry transformation

$$\phi_k\to\phi_k+i\chi T_{kl}\phi_l, \tag{7.39}$$

where T_{kl} are numbers defining the symmetry and $\chi\ll 1$ is an infinitesimal parameter; for mathematically minded, T is the generator of the symmetry algebra. The infinitesimal change induced into the Lagrangian by the symmetry transformation is

$$\begin{aligned}\delta L&=\frac{\partial L}{\partial\phi_k}\delta\phi_k+\frac{\partial L}{\partial\partial_i\phi_k}\partial_i\delta\phi_k\\&=(\frac{\partial L}{\partial\phi_k}-\partial_i\frac{\partial L}{\partial\partial_i\phi_k})\delta\phi_k+\partial_i[\frac{\partial L}{\partial\partial_i\phi_k}\cdot\underbrace{\delta\phi_k}_{\sim T_{kl}\phi_l}].\end{aligned}\tag{7.40}$$

The last term is a surface one, which is neglected when deriving the EOM. However, if now $\phi_k^{\mathrm{extr}}(\boldsymbol{x})$ satisfies the classical EOM, so that

$$\frac{\partial L}{\partial\phi_k}-\partial_i\frac{\partial L}{\partial\partial_i\phi_k}=0\ ,$$

requiring invariance $\delta L=0$ we can conclude that

$$\partial_i\mathcal{J}_i=0, \tag{7.41}$$

where

$$\mathcal{J}_i=\frac{\partial L}{\partial\partial_i\phi_k}\cdot T_{kl}\phi_l\bigg|_{\phi_k=\phi_k^{\mathrm{extr}}} \tag{7.42}$$

is the *Noether current*.

For global U(1) symmetry the infinitesimal transformation is

$$M \to M + i\chi M, \quad M^* \to M^* - i\chi M^*. \tag{7.43}$$

Since the partial derivative needed in Eq. (7.42) is

$$\frac{\partial L}{\partial \partial_i M} = K \partial_i M^* \tag{7.44}$$

we have the symmetry current for the global U(1) symmetry:

$$\mathcal{J}_i = iK[\partial_i M^* \cdot M - \partial_i M \cdot M^*] = -2K\,\mathrm{Im}(\partial_i M^* \cdot M) \tag{7.45}$$

which for the vortex ansatz $M = f(r)e^{i\phi}$ becomes

$$= -2K\,\mathrm{Im}(\partial_i f \cdot e^{-i\phi} - i\partial_i \phi \cdot f e^{-i\phi}) f e^{i\phi} \; = 2K f^2(r) \partial_i \phi(x, y). \tag{7.46}$$

Note the dimensions: $[\boldsymbol{\mathcal{J}}] = [T/x^2]$ = energy/area.

7.5 Ginzburg-Landau theory of superconductivity

Let us now make the U(1) symmetry in Eq. (7.27) *local*, i.e., demand that the free energy (or action) be invariant under the **x**-dependent transformation

$$M \equiv \phi(\boldsymbol{x}) \to e^{i\chi(\boldsymbol{x})}\phi(\boldsymbol{x}) \simeq (1 + \chi(\boldsymbol{x}))\phi(\boldsymbol{x}) \; . \tag{7.47}$$

$V(\phi^*\phi)$ is still invariant but $\partial_i \phi^* \, \partial_i \phi$ is not. However, if we replace ∂_i by the covariant derivative

$$\partial_i \to \partial_i + ie_3 A_i \equiv D_i \; , \tag{7.48}$$

where we have introduced a gauge field $A_i(\boldsymbol{x})$ which is required to transform as

$$A_i \to A_i - \frac{1}{e_3}\partial_i \chi \qquad (\partial_i \equiv \frac{\partial}{\partial x^i} \equiv \nabla_i) \; , \tag{7.49}$$

then

$$\begin{aligned} D_i\phi(\mathbf{x}) &\to [\partial_i + ie_3(A_i - \frac{1}{e_3}\partial_i\chi)]e^{i\chi(\boldsymbol{x})}\phi(\boldsymbol{x}) \\ &= i\partial_i\chi \cdot e^{i\chi}\phi + e^{i\chi}\partial_i\phi + ie_3 A_i \cdot e^{i\chi}\phi - i\partial_i\chi \cdot e^{i\chi}\phi \\ &= e^{i\chi} D_i \phi(\mathbf{x}) \; . \end{aligned} \tag{7.50}$$

The covariant derivative transforms covariantly, i.e. as a group element like $\phi(\boldsymbol{x})$. Hence the kinetic term $(D_i\phi)^* \, D_i\phi$ *is* invariant even under local U(1) transformations.

The kinetic part of the gauge field ($\sim \partial^2 A^2$) should also be invariant and, of course, we know it is just the Maxwell action of electrodynamics

$$\sum_{i,j}^{3} F_{ij}^2 = \sum_{i,j=1}^{3} \frac{1}{4}(\partial_i A_j - \partial_j A_i)(\partial_i A_j - \partial_j A_i) \equiv \frac{1}{2}\boldsymbol{B}^2, \tag{7.51}$$

where, in 3 dimensions, we only need the magnetic field

$$B_1 = \partial_2 A_3 - \partial_3 A_2 \ldots, \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{7.52}$$

Thus we have been led to a theory defined by the functional integral

$$Z \equiv e^{-Vf} = \int \mathcal{D}A_i(\boldsymbol{x})\, \mathcal{D}\phi(\boldsymbol{x})\, e^{-S[A_i(\mathbf{x}),\phi(\mathbf{x})]}\ , \tag{7.53}$$

where

$$\begin{aligned} S[A_i(\mathbf{x}),\phi(\mathbf{x})] &\equiv \frac{1}{T} F[A_i(\mathbf{x}),\phi(\mathbf{x})] = \\ &= \frac{1}{T}\int d^3x \Big[\frac{\tilde{K}}{4}(\partial_i A_j - \partial_j A_i)^2 + K|(\partial_i + ie_3 A_i)\phi|^2 + a|\phi|^2 + b|\phi|^4\Big] \end{aligned} \tag{7.54}$$

In general, this is 3d scalar electrodynamics; i.e., 3d gauge + scalar field theory. It is a superrenormalisable field theory (only two divergent diagrams, the tadpole and the sunset diagrams). In terms of this general formulation, all physics is contained in the expectation values of correlators of gauge invariant operators:

$$\langle O_1(\mathbf{x}_1)...O_n(\mathbf{x}_n)\rangle = \frac{1}{Z}\int \mathcal{D}A_i(\boldsymbol{x})\, \mathcal{D}\phi(\boldsymbol{x})\, O_1(\mathbf{x}_1)...O_n(\mathbf{x}_n) e^{-S[A_i(\mathbf{x}),\phi(\mathbf{x})]}, \tag{7.55}$$

where the O_i are local gauge invariant operators of the type

$$O(\mathbf{x}) = F_{ij}^2(\mathbf{x}), \quad |\phi(\mathbf{x})|^2, \quad \phi^*(\mathbf{x})D_i\phi(\mathbf{x}), \ldots. \tag{7.56}$$

For the time being we shall discuss this theory only in the mean field approximation, i.e., study the minima of the free energy in Eq. (7.54). This approximation is appropriate for superconductivity.

For the Ginzburg-Landau theory of superconductivity the parameters in Eq. (7.54) are given by

$$\widetilde{K} = \frac{1}{\mu_0}, \quad e_3 = \frac{e}{\hbar}, \quad K = \frac{\hbar^2}{2m} \tag{7.57}$$

and further also $e \to 2e$ and $m \to 2m_e^*$, the charge and mass of Cooper pairs, respectively. In the Ginzburg-Landau theory the complex scalar field $\phi(\boldsymbol{x})$ is interpreted as a "condensate wave function" and its absolute value squared, $|\phi(\boldsymbol{x})|^2$ as the "density of Cooper pairs", pairs of e^- near Fermi surface. If $a < 0$, the U(1) symmetry is broken and an expectation value of $\phi^*\phi$ appears. As a consequence, the system goes over into a superconducting (SC) phase (note: there is so far no mention of persistent electric currents!).

For bulk superconductivity again the gradient terms are $= 0$ (if nothing forces A_i, ϕ to vary, they remain constant). For inhomogenous situations (boundaries, external B) they are essential, and then, for dimensional reasons, two distance scales appear in the broken (SC) phase:

- For ϕ:

$$K\partial^2\phi^2 + a\phi^2 \quad \Rightarrow \xi = \sqrt{\frac{K}{-a}} \tag{7.58}$$

which as before is the coherence length for the "kink" $= \sqrt{2}/m_{\text{Higgs}}$.

- For A_i:

$$\begin{aligned} \widetilde{K}\partial^2 A^2 + \overbrace{K\, e_3^2\, \phi^2}^{\text{"photon mass"}} \;\; A^2 \Rightarrow \delta \;&=\; \sqrt{\frac{\widetilde{K}}{K}}\frac{1}{e_3 < |\phi|\sqrt{2} >} \\ &=\; \sqrt{\frac{\widetilde{K}}{K}}\frac{1}{e_3\sqrt{\frac{-a}{b}}} = \frac{\hbar}{m_\gamma c} \end{aligned} \tag{7.59}$$

which is the penetration depth of $\boldsymbol{A}$ and $\boldsymbol{B}$.

The ratio of the two scales, κ, is an important parameter:

$$\kappa^2 = \frac{\delta^2}{\xi^2} = \frac{\widetilde{K}}{K}\frac{b}{e_3^2} \ . \tag{7.60}$$

The value $\kappa^2 = 1/2$ separates type I and type II superconductors, with significantly different properties.

7.6 London equation

The classical field configuration ϕ, A_k again has to satisfy the equations of motion. For ϕ one varies F with respect to ϕ^* and obtains the EOM

$$-KD_kD_k\phi + a\phi + 2b|\phi|^2\phi = 0. \tag{7.61}$$

This is just (7.28) with $\partial_i \to D_i$.

For A_k we have

$$\frac{\delta F}{\delta A_k} = 0 \Rightarrow \frac{\partial L}{\partial A_k} - \partial_i \frac{\partial L}{\partial\partial_i A_k} = 0 \ , \tag{7.62}$$

which leads to

$$-\widetilde{K}\partial_i[\partial_i A_k - \partial_k A_i] - iKe_3\left[\phi^* D_k\phi - (D_k\phi)^*\phi\right] = 0 \ , \tag{7.63}$$

where $-\widetilde{K}\partial_i[\partial_i A_k - \partial_k A_i] = \widetilde{K}(\boldsymbol{\nabla}\times\boldsymbol{B})_k$ This, of course, is just Maxwell's equation ($\widetilde{K} \to \mu_0$) $\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\mathbf{J}$ without the electric field term and with a derived current satisfying current conservation:

$$\begin{aligned} J_k &= +iKe_3\big[\phi^*(\partial_k\phi + ie_3A_k\phi) - (\partial_k\phi^* - ie_3A_k\phi^*)\phi\big], \\ \partial_k J_k &= \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0 \ . \end{aligned} \tag{7.64}$$

This is just the current (7.45) with the derivative ∂_i replaced by the covariant derivative D_i. Inserting here $\phi = |\phi| e^{iS}$, one obtains the "London equation" (1935)

$$\boldsymbol{J} = -2Ke_3|\phi|^2\boldsymbol{\nabla}S - 2e_3^2K|\phi|^2\boldsymbol{A} = \text{electric current} = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\boldsymbol{B}\ , \tag{7.65}$$

where $\boldsymbol{\nabla}S$ is the gradient of phase. The current $\mathbf{J}$ is, by construction, locally gauge invariant. Gauge invariance requires both the phase S and $\boldsymbol{A}$; a change in S is compensated for by a change in $\mathbf{A}$. For a simply connected region one can "gauge away" the phase, just do the gauge transformation $\phi \to e^{-iS(\boldsymbol{x})}\phi$. To see the reason for this limitation, take S to be the azimuthal angle around the z axis, $\tan S = y/x$. Then $\boldsymbol{\nabla}\times\boldsymbol{\nabla}S = 2\pi\delta^2(\mathbf{r}),\ \mathbf{r} = (x, y, 0)$ (cf. the discussion of superfluid rotational flow) and ($d\mathbf{A}$ is here the surface element)

$$\oint \nabla S \cdot d\mathbf{l} = \int \nabla \times \nabla S \cdot d\mathbf{A} = 2\pi.$$

Thus the existence of the vortex line prevents one from gauging away S.

So where is the magic supercurrent, current without ΔV? The key is symmetry breaking, we have

$$|\phi|^2 = \begin{cases} 0 \text{ in symmetrical normal phase.} \\ \frac{-a}{2b} \text{ in broken SC phase.} \end{cases} \tag{7.66}$$

so that the current vanishes in the normal symmetric phase but it is non-zero in the broken phase. Introducing the penetration length by

$$\frac{1}{\delta^2} = \frac{2Ke_3^2|\phi|^2}{\widetilde{K}} \tag{7.67}$$

and assuming that $|\phi| = $ constant, we can as well write the London equation (7.65) in the form

$$\delta^2\boldsymbol{\nabla}\times\boldsymbol{B} + \boldsymbol{A} = -\frac{2\pi}{e_3}\frac{\boldsymbol{\nabla}S}{2\pi}\ , \tag{7.68}$$

where

$$\frac{2\pi}{e_3} = \frac{2\pi\hbar}{e} = \frac{h}{e} = 4.13\cdot 10^{-15}\text{Vs} \tag{7.69}$$

is the flux quantum. Taking again S to be the azimuthal angle and applying $\boldsymbol{\nabla}\times$ once more leads to

$$\delta^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B} = -\delta^2\nabla^2\boldsymbol{B} + \boldsymbol{B} = -\frac{2\pi}{e_3}\delta^2(\mathbf{r}). \tag{7.70}$$

To see the meaning of δ, choose a gauge in which $S = 0$ and the geometry $\boldsymbol{A} = (0,\ A(x),\ 0) \Rightarrow \boldsymbol{B} = (0,\ 0,\ A'(x)) \Rightarrow \boldsymbol{\nabla}\times\boldsymbol{B} = (0, -A''(x), 0)$. Then

$$-\delta^2A''(x) + A(x) = 0 \quad \Rightarrow A(x) = e^{-x/\delta}A(0)\ . \tag{7.71}$$

Thus the supercurrent is located near the surface, together with the magnetic field B. This is the Meissner effect. We shall study this with more realistic approximations in Section 7.8.

7.7 Ginzburg-Landau as 3d field theory

The three-dimensional U(1)+Higgs theory is a locally gauge invariant 3-dimensional continuum U(1) + complex scalar field theory defined by the functional integral

$$Z = \int \mathcal{D}A_i \mathcal{D}\phi \exp\big[-S(A_i,\phi)\big] = \exp\big[-Ve_3^6 f(y,x)\big], \tag{7.72}$$

$$S = \int d^3x \left[\frac{1}{4}(\partial_i A_j - \partial_j A_i)^2 + |(\partial_i + ie_3 A_i)\phi|^2 + m_3^2\phi^*\phi + \lambda_3\,(\phi^*\phi)^2\right]. \tag{7.73}$$

The parameters m_3, e_3^2, λ_3 of the Lagrangian have the dimension GeV and the fields have dimension GeV$^{1/2}$. Since the theory in eq. (7.73) is a continuum field theory, one has to carry out ultraviolet renormalization. In 3d the couplings e_3^2 and λ_3 are not renormalised in the ultraviolet, but there is a linear 1-loop and a logarithmic 2-loop divergence for the mass parameter m_3^2. In the $\overline{\text{MS}}$ dimensional regularization scheme in $3-2\epsilon$ dimensions the renormalized mass parameter becomes

$$m_3^2(\mu) = \frac{-4e_3^4 + 8\lambda_3 e_3^2 - 8\lambda_3^2}{16\pi^2}\log\frac{\Lambda_m}{\mu}, \tag{7.74}$$

where μ is the running scale and Λ_m is a scale independent physical mass parameter of the theory. Instead of it it is more convenient to use $m_3(e_3^2)$. Choosing e_3^2 to set the scale, the physics of the theory will depend on the two dimensionless ratios

$$y = \frac{m_3^2(e_3^2)}{e_3^4}, \quad x = \frac{\lambda_3}{e_3^2}. \tag{7.75}$$

The standard tree-level symmetry breaking analysis starts by inserting to the action $\phi = (v + \phi_1 + i\phi_2)/\sqrt{2}$ (leave out the subscript 3):

$$\begin{aligned} S = \int d^3x \Big[& \tfrac14(\partial_i A_j - \partial_j A_i)^2 + \tfrac12 e^2(v^2 + 2v\phi_1 + \phi_1^2 + \phi_2^2)A_iA_i + evA_i\partial_i\phi_2 + eA_i(\phi_1\partial_i\phi_2 - \phi 2\partial_i\phi_1) \\ & + \tfrac12 m^2v^2 + \tfrac14\lambda v^4 + \tfrac12(\partial_i\phi_1\partial_i\phi_1 + \partial_i\phi_2\partial_i\phi_2) + \tfrac12(m^2 + 3\lambda v^2)\phi_1^2 + \tfrac12(m^2+\lambda v^2)\phi_2^2 \\ & + (m^2 + \lambda v^2)v\phi_1 + \tfrac14\lambda(\phi_1^2+\phi_2^2)^2 + \lambda v\phi_1((\phi_1^2+\phi_2^2)\Big], \end{aligned} \tag{7.76}$$

from which one reads the usual tree-level ground state values

$$v = \sqrt{\frac{-m^2}{\lambda}}, \quad m_1 = m_{\text{Higgs}} = \sqrt{-2m^2} = \sqrt{2\lambda}v, \quad m_2 = m_{\text{Goldstone}} = 0, \quad m_A = ev. \tag{7.77}$$

Earlier $\xi = \sqrt{2}/m_{\text{Higgs}}$ was the correlation length and $\delta = 1/m_A$ was the penetration length. There are many different names and notations for these two quantities. On the tree level

$$\frac{m_H^2}{m_A^2} = \frac{2\lambda}{e^2} = 2\kappa^2 = 2x;$$

large Higgs mass correponds to large λ (as also in the Standard Model) and the tree level boundary between type I and II is at $x = \kappa^2 = 1/2$.

Symmetry breaking has also produced several new interaction terms in Eq. (7.76). The mixed term $m_A A_i\partial_i\phi_2$ can be disposed of by partially integrating and choosing the Coulomb gauge $\partial_i A_i = 0$. Further perturbative quantisation of the theory Eq. (7.76) by gauge fixing is discussed in particle physics text books[1]. However, perturbation theory has only a limited range of applicability and, fundamentally, all physics lies in expectation values of various operators. Since this is a gauge theory, only gauge invariant operators have non-vanishing expectation values. The most relevant one of these are the local (depending only on one point x) operators of lowest dimensionality:

[1]See e.g. Bailin-Love, Introduction to gauge field theory, section 13.5

- Dim = 1: the $J^{PC} = 0^{++}$ scalar $O(\mathbf{x}) = \phi^\star(\mathbf{x})\phi(\mathbf{x})$,
- Dim = 1.5: the $J^{PC} = 1^{+-}$ vector $\tilde{O}_i(\mathbf{x}) \equiv B_i = \epsilon_{ijk}F_{jk}(\mathbf{x})/2$ ($F_{ij} = \partial_i A_j - \partial_j A_i$),
- Dim = 2: the $J^{PC} = 1^{--}$ vector $O_i(\mathbf{x}) = \mathrm{Im}\phi^\star(\mathbf{x})D_i\phi(\mathbf{x})$ ($D_i = \partial_i + ie_3 A_i$), the 1^{-+} vector $\mathrm{Re}\phi^\star D_i\phi = \partial_i\phi^\star\phi/2$ and the 0^{++} scalar $(\phi^\star\phi)^2$,
- Dim=3: The 0^{++} scalars $F_{ij}F_{ij}$ and $\phi^\star D_i D_i\phi$, the 1^{+-} vector $\phi^\star B_i\phi$ and the 2^{++} tensor $\phi^\star[\{D_i, D_j\} - 2/d\ \delta_{ij}D_k D_k]\phi$,
- Dim = 3.5: The 0^{--} scalar $B_i\,\partial_i\phi^\star\phi$ and the 0^{-+} scalar $B_i\,\mathrm{Im}\phi^\star D_i\phi$,
- Dim = 4: The 0^{+-} scalar $\partial_i\phi^\star\phi\,\mathrm{Im}\phi^\star D_i\phi$

The quantum numbers here refer to O(3). From these one can further construct bilocal, etc. operators and correlators of the above operators, depending on two points.

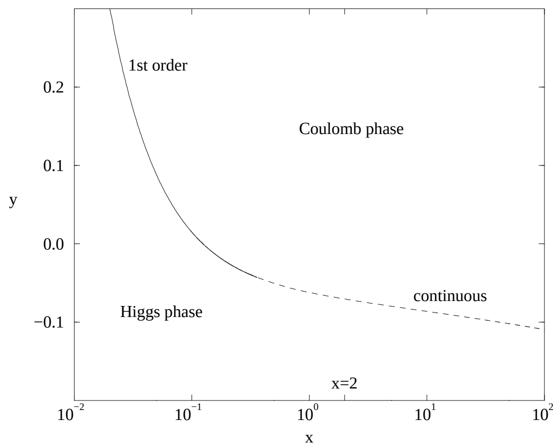

Figure 7.8: The phase diagram of 3d GL theory. Order parameters separating the phases are either the gauge field mass or the vortex tension.

The first topic of importance is the phase structure of the theory. The critical curve $y = y_c(x)$ (see Fig. 7.7) divides the plane in two disjoint regions, the symmetric phase at $y > y_c(x)$ and the broken phase at $y < y_c(x)$. The presence of a critical curve is signalled by singularities in the free energy $Z = \exp\big[-Ve_3^6 f(y,x)\big]$. On the tree level the critical curve is at $m_3^2 = 0$ (or at $y = 0$). Including fluctuations to one loop, the effective potential becomes

$$V_{1-\mathrm{loop}}/e_3^6 = \frac{1}{4}x\hat{\phi}^2\left[\left(\hat{\phi} - \frac{1}{3\pi x}\right)^2 + \frac{2y}{x}\left(1 - \frac{1}{18\pi^2 xy}\right)\right]. \tag{7.78}$$

Two degenerate states are obtained when the last term vanishes. From this one finds for the critical curve $y_c(x)$, for the upper and lower metastability branches $y_\pm(x)$, the latent heat-like jump $\Delta\ell_3$ of the order parameter-like quantity $\ell_3 \equiv \langle\phi^\dagger\phi(e_3^2)\rangle/e_3^2$ between the broken and symmetric phases at y_c and for the interface tension $\sigma_3 \equiv \sigma/e_3^4$, defined in perturbation theory by

$$\sigma_3 = \int_0^{\phi_b/e_3} d(\phi/e_3)\sqrt{2V(\phi/e_3)/e_3^6}, \tag{7.79}$$

where V is the perturbatively computed effective potential, the following values

$$y_c(x) = \frac{1}{18\pi^2 x}, \quad y_+(x) = \frac{1}{16\pi^2 x}, \quad y_-(x) = 0, \tag{7.80}$$

$$\hat{\phi}_{\rm symm} = 0, \quad \hat{\phi}_{\rm broken} = \frac{1}{3\pi x}, \tag{7.81}$$

$$\ell_3 = \frac{1}{18\pi^2 x^2}, \quad \sigma_3 = \frac{2^{3/2}}{648\pi^3 x^{5/2}}. \tag{7.82}$$

Perturbation theory becomes unreliable for large λ_3 or $x > 0.1$ and the transition becomes continuous there. This is also the region of type II superconductors. What is crucial is that there **is** a phase transition there, the phase transition does not first have a first order line which then ends in a second order critical point like the Ising model transition. The point is that there is an order parameter which distinguishes these two phases. This order parameter could, for example, be the gauge field mass. $m_A \equiv m_V \equiv m_\gamma$, which is $= 0$ in the symmetric Coulomb and $\neq 0$ in the broken Higgs phase. Equally, it could be the tension of a vortex line. This can be calculated in the mean-field approximation by solving the field equations numerically. The result can be written in the form

$$T_{\rm MF} = \frac{\Delta S}{L} = -\frac{y}{x}\pi\mathcal{E}(\sqrt{2x}), \tag{7.83}$$

where the function $\mathcal{E}$, with the value $\mathcal{E}(1) = 1$, has been calculated numerically in, e.g., Ref. [**?**]. Of course, one must also be able to compute it quite generally as an operator expectation value. Much work on this has been done in Helsinki. **Referenssit**

If one extends the above from U(1)+Higgs gauge theory to SU(2)+fundamental representation Higgs gauge theory, one obtains a theory which describes the thermodynamics of the electroweak theory. Then the phase diagram looks much like that in Fig. 7.7 but with the qualitatively crucial difference: there the 1st order line really ends in a critical point, for larger x or self-coupling λ or the Higgs mass there is no phase transition, only a "cross-over". When all numbers are put carefully together, the endpoint corresponds to Higgs masses of the order of 77 GeV. This is far below the present (2006) lower limit of m_H so that in the physical minimal standard model there is no electroweak phase transition.

7.8 Normal-superconducting interface tension

Consider now a physical situation in which one forces a magnetic field through a superconductor.[2] Superconductivity then must be at least locally lost and superconductors differ in how this happens. For type I B penetrates through a thick rope while for type II B penetrates through a lattice of vortices. Effectively, in type I the vortices attract each other so that it is more favourable to join them in a rope. In type II they repel and want to be as far from each other as possible, i.e., they form a lattice. Quantitatively, an elegant way to see this difference is to compute the interface tension σ of a planar interface between normal and superconducting phases. For positive σ it pays to reduce the number of interfaces so that the vortices attract each other.

Assume now that the curvature of the interface in Fig. 7.8 is so small that it can be taken as planar. Assume further that the interface is in the y, z plane and that the magnetic and the

[2] Landau & Lifshitz Stat Phys vol. 2 §46

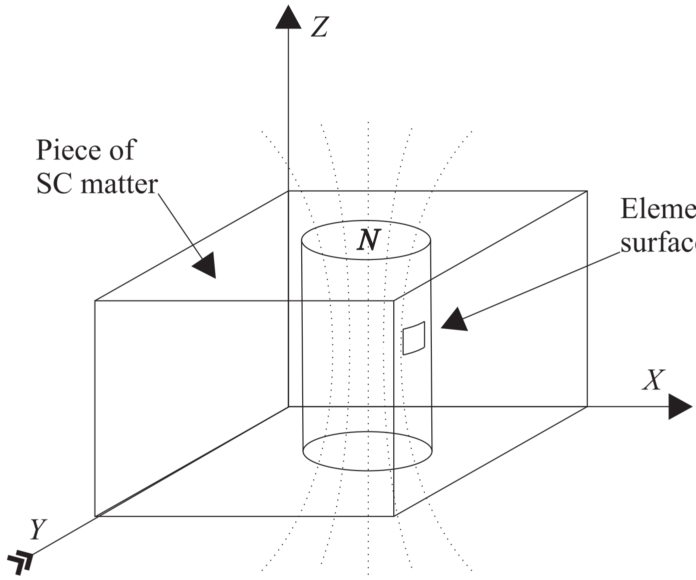

Figure 7.9: What happens if you force a magnetic field through a superconductor? For type I a rope is formed.

order parameter fields only depend on x and are[3]

$$\phi = \phi(x) = \text{real} \quad \text{(gauge choice)}, \tag{7.84}$$
$$\boldsymbol{A} = (0,\, A(x),\, 0), \tag{7.85}$$
$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = (0,\, 0,\, A'(x)). \tag{7.86}$$

The gauge has been fixed by transforming away the phase of ϕ. We also change the notation to the particle physics one, $a \to m^2$, $b \to \lambda$, $K = \widetilde{K} = 1$. Note that for the geometry of Fig. 7.8 we want $\mathbf{B}$ to be parallel to the interface. The interface tension is F/Area, where

$$F = \int \underbrace{dy\, dz}_{=\text{area}}\, dx \left[\frac{1}{4}F_{ij}^2 + (D_i\phi)^* D_i\phi + m^2\phi^*\phi + \lambda(\phi^*\phi)^2\right].$$

For the present interface geometry

$$\begin{aligned} \tfrac{1}{4}\, F_{ij}^2 &= \tfrac{1}{2}\, B_z^2 = \tfrac{1}{2}\left[A'(x)\right]^2, \\ |D_i\phi|^2 &= (\partial_i\phi - ie_3A_i\phi)(\partial_i\phi + ie_3A_i\phi) = [\phi'(x)]^2 + e_3^2A^2\phi^2, \end{aligned}$$

so that

$$\frac{F}{\text{Area}} = \int_{-\infty}^{\infty} dx \Big[\frac{1}{2}(A'(x))^2 + (\phi'(x))^2 + (e_3^2A^2(x) + m^2)\phi^2(x) + \lambda\phi^4(x)\Big] \equiv \int_{-\infty}^{\infty} dx f(x). \tag{7.87}$$

The equations of motion extremising this are

$$\begin{aligned} \tfrac{\partial L}{\partial A} - \partial_x \tfrac{\partial L}{\partial A'} = 0 \qquad &\Rightarrow \qquad A''(x) = 2e_3^2A(x)\phi^2(x) \\ \tfrac{\partial L}{\partial \phi} - \partial_x \tfrac{\partial L}{\partial \phi'} = 0, \qquad &\Rightarrow \qquad \phi''(x) = \left[e_3^2A^2(x) + m^2\right]\phi(x) + 2\lambda\phi^3(x)\ . \end{aligned} \tag{7.88}$$

[3] A typical source of confusion, if one wants to compare with some earlier equations for the scalar field, is that we here have not included a factor $1/\sqrt{2}$ in $\phi(x)$.

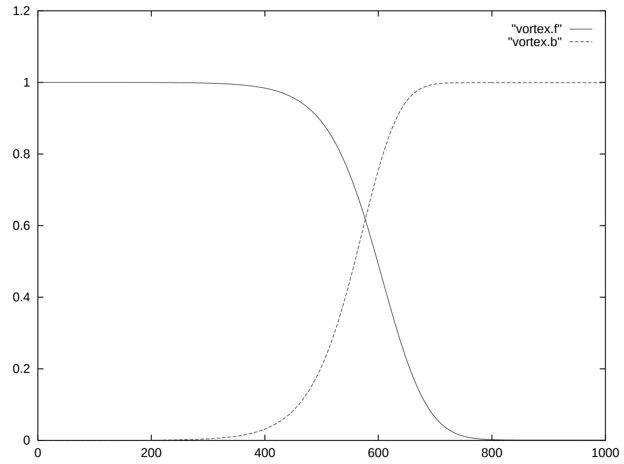

Figure 7.10: Configurations of $\phi(x)$ (grows to the left) and $B(x)$ (grows to the right) for zero interface tension, $\lambda = \frac{1}{2}e_3^2$. Here the SC broken phase is to the left, $x < 0$.

To solve these equations of motion we also need boundary conditions. We fix them so that the SC phase is at $x > 0$ and the normal phase at $x < 0$. Then, for $x \to \infty$, $\phi(x) \to \langle\phi\rangle = \sqrt{-m^2/2\lambda}$, $A \to 0$, $f(x) \to -m^4/(4\lambda)$. For $x \to -\infty$ $B = A' \to H_c$, $\phi \to 0$, $f(x) \to \frac{1}{2}H_c^2$, where the limiting value H_c will be determined in a moment. To isolate $f_{\rm interface}(x)$ we must demand that $f_{\rm interface}(x) \to 0$ for $x \to \pm\infty$. This is solved for $x \to +\infty$ by modifying $f(x)$ by $(A')^2 \to (A' - H_c)^2$:

$$\frac{F_{\rm interface}}{\rm Area} = \sigma = \int_{-\infty}^{\infty} dx \Big[\frac{1}{2}(A' - H_c)^2 + (\phi')^2 + (e_3^2A^2 + m^2)\phi^2 + \lambda\phi^4\Big]. \qquad (7.89)$$

so that

$$f_{\rm interface}(x \to +\infty) = \frac{1}{2}H_c^2 - \frac{m^4}{4\lambda} = 0. \qquad (7.90)$$

Thus one has determined

$$\frac{1}{2}H_c^2 = \frac{m^4}{4\lambda}, \qquad (7.91)$$

which just expresses the fact that the magnetic energy density on the normal side should equal the condensate energy density on the broken side.

Multiplying the first of the EOMs (7.88) by A' and the second by ϕ' one derives a first integral

$$\frac{1}{2}A'^2 - (e_3^2A^2 + m^2)\phi^2 + \phi'^2 - \lambda\phi^4 = \frac{m^4}{4\lambda}, \qquad (7.92)$$

where the constant is determined from boundary conditions. With this one can further write the interface tension in equivalent forms

$$\sigma = \int_{-\infty}^{\infty} dx \left(A'^2 - H_c A' + 2\phi'^2\right) = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(A' - H_c)^2 - \lambda\phi^4\right]. \qquad (7.93)$$

From this we can analytically compute that

- For small $\lambda \ll e_3^2$ the coherence length $\xi = 1/\sqrt{-m^2}$ is much larger than the penetration length $\delta = \sqrt{\lambda/e_3^2(-m^2)}$. The magnetic field then drops suddenly at the interface at $x = 0$ to zero and we can focus only on ϕ. Eq. (7.92) then becomes $\phi' = \sqrt{\lambda}(\langle\phi\rangle^2 - \phi^2)$ which is solved by $\phi = \langle\phi\rangle \tanh(x/\sqrt{2}\xi)$. Inserting this to the first of (7.93) then gives, integrating over $x > 0$,
$$\sigma = \frac{\sqrt{2}}{3}\frac{(-m^2)^{3/2}}{\lambda}. \qquad (7.94)$$
- The tension decreases with increasing λ and, remarkably, it vanishes, according to the second of (7.93), if
$$A' - H_c = -\sqrt{2\lambda}\phi^2. \qquad (7.95)$$
A closer inspection of the EOMs shows that this happens when $\lambda = \frac{1}{2}e_3^2$ or $\kappa^2 = \frac{\delta^2}{\xi^2} = \frac{\widetilde{K}}{K}\frac{b}{e_3^2} = \frac{1}{2}$. This is shown in detail in Eqs.(46.15-46.18) of Landau & Lifshiz, Stat Phys, part 2, §46. The corresponding numerically computed field configurations are shown in Fig. 7.8.
- For $\lambda > \frac{1}{2}e_3^2$ $\sigma < 0$ and one minimises free energy by maximising the amount of interface, i.e., it pays to distribute the total amount of flux among a number of vortex lines, each carrying a flux quantum h/e.

We thus have two types of superconductors, which differ in how they react to an imposed flux of magnetic field:

$$\Phi_B = B \cdot \text{Area}. \qquad (7.96)$$

- Type I has $\sigma > 0$ and the imposed flux forms a rope parallel to $\mathbf{B}$ to minimize the area (perpendicular to $\mathbf{B}$).
- Type II has $\sigma < 0$ and the imposed flux penetrates the plasma through a lattice of (Abrikosov) vortices parallel to $\mathbf{B}$.

Thermodynamically, B is a "canonical" variable (or actually its extensive volume integral $VB = L_z \cdot \Phi_B$), which can be spatially inhomogeneous. Its analogue is N, the particle number. For B we can write $VB = L_z(2\pi/e_3)N$, where N now is the number of flux quanta. The conjugate intensive "grand canonical" (it lets $\Phi_B = BA$ or the number of flux quanta fluctuate) variable is H, which is homogeneous, like the chemical potential μ. Thus we can write $F(T, V, N, B) = E - TS = -pV + \mu N + H \cdot VB$, taking account of the vector nature of $\mathbf{B}$ in a suitable way.