Stochastic processes. Wiener path integral

Stochastic process - random process (a collection of random variables.

As stochastic process can be considered any time evolution that can be described by PROBABILITIES.

Brownian motion (Robert Brown, 1828) - the simplest example of a stochastic process

- classical systems subject to random influences from the surronnding medium.
- mathematical theory of Brownian motion:

A. Einstein (1905) (Ann. der Physik 17 (1905) 549)

M. Smoluchowski (1906)

Brownian motion of a free particle. Diffusion equation

- Consider a large number of particles (N) performing Brownian motion
- Take, for simplicity, one-dimensional motion along a given axis

$\rho(x,t)$ - density of particles at time t

$j(x,t)$ - current of particles: the number of Brownian particles passing the point x in the direction of increasing values of x per unit time

- experimentally known:

$$j(x,t) = -D \frac{\partial \rho(x,t)}{\partial x} \qquad (1)$$

- continuity equation (particles are not created, nor destroyed):

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x} \qquad (2)$$

- from (1) and (2) follows the equation describing the behaviour of Brownian particles :

$$\frac{\partial \rho(x,t)}{\partial t} = D \frac{\partial^2 \rho(x,t)}{\partial x^2}, \qquad (3)$$

D - diffusion constant

DIFFUSION EQUATION

- $w(x,t)$ - probability distribution

$$\rho(x,t) = N\, w(x,t),$$

N - total number of Brownian particles

$x \in \mathbb{R}$;
$t \geq 0$.

- Solution of the diffusion equation
  - initial condition

$$w(x,t) \xrightarrow[t \to 0]{} \delta(x)$$

→ Green function (fundamental solution) of diffusion eq.

$$\frac{\partial w(x,t)}{\partial t} = D \frac{\partial^2 w(x,t)}{\partial^2 x} \qquad (4)$$

- Fourier transform

$$w(x,t) = \int_{-\infty}^{\infty} \tilde{w}(k,t)\, e^{ikx}\, dk \qquad (5)$$

$$w(x,0) = \int_{-\infty}^{\infty} \tilde{w}(k,0)\, e^{ikx}\, dk \qquad (6)$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk \qquad (7)$$

$$w(x,0) = \delta(x) \Downarrow \tilde{w}(k,0) = \frac{1}{2\pi}$$

- Introduce (5) into (4)

$$\left.\begin{aligned} \dot{\tilde{w}}(k,t) &= -D k^2 \tilde{w}(k,t) \\ \tilde{w}(k,0) &= \frac{1}{2\pi} \end{aligned}\right\} \Rightarrow \tilde{w}(k,t) = w(k,0)\, e^{-Dk^2 t} = \frac{1}{2\pi} e^{-Dk^2 t} \qquad (8)$$

$$w(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dk^2t}\, e^{ikx}\, dk \qquad (9)$$

- to perform the integral over k, use the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \qquad (10)$$

$$e^{-Dk^2t}\, e^{ikx} = e^{-Dt\left(k^2 - \frac{i}{Dt}kx\right)} =$$
$$= e^{-Dt\left(k^2 - 2\frac{i}{2Dt}kx - \frac{x^2}{4D^2t^2}\right)}\, e^{-\frac{x^2}{4Dt}} = e^{-Dt\left(k - \frac{ix}{2Dt}\right)^2}\, e^{-\frac{x^2}{4Dt}} \qquad (11)$$

- from (9) and (11), using (10):

$$\boxed{w(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-\frac{x^2}{4Dt}}} \qquad (12)$$

$w(x,t)$ – probability of finding the Brownian particle at the place x at the moment t, if the particle was at the origin $x=0$ at the time $t=0$.

$\Downarrow$ necessarily

$$\int_{-\infty}^{\infty} w(x,t)\, dx = 1 \qquad (13)$$

Indeed,

$$\int_{-\infty}^{\infty} w(x,t)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}}\, e^{-\frac{x^2}{4Dt}} dx = 1,$$

what is compatible with the above probability interpretation.

- $W(x_t, t \mid x_0, t_0)$, $x_t = x(t)$, $x_0 = x(t_0)$ TRANSITION PROBABILITY, characterizing the time evolution of the probability density, i.e.

$$w(x_t, t) = \int_{-\infty}^{\infty} W(x_t, t \mid x_0, t_0)\, w(x_0, t_0)\, dx_0, \qquad (14)$$

with $w(x_0, t_0)$ an arbitrary function, satisfying the normalization condition (13).

- from (14) and (4) it follows that

$$\frac{\partial W(x_t, t \mid x_0, t_0)}{\partial t} = D \frac{\partial^2 W(x_t, t \mid x_0, t_0)}{\partial x_t^2}, \quad t_0 < t \qquad (15)$$

with initial condition

$$W(x_t, t \mid x_0, t_0) \xrightarrow[t \to t_0]{} \delta(x_t - x_0) \qquad (16)$$

Then

$$W(x_t, t \mid x_0, t_0) = \frac{1}{\sqrt{4\pi D(t-t_0)}} e^{-\frac{(x_t - x_0)^2}{4D(t-t_0)}} \qquad (17)$$

and satisfies the normalization

$$\int_{-\infty}^{\infty} dx_t \, W(x_t, t \mid x_0, t_0) = 1. \qquad (18)$$

$W(x_t, t \mid x_0, t_0)$ – normal (Gaussian) probability distribution with mean value x_0 and dispersion $2D(t-t_0)$

<u>Properties of the Brownian motion</u>

i) HOMOGENEITY IN SPACE – transition probability is function only of $x_t - x_0$;

ii) HOMOGENEITY in TIME – transition probability does not depend on the initial moment, but only on the difference $t - t_0$;

iii) ISOTROPY – transition probability does not depend on the direction in space, i.e. is left invariant if we change x_t, x_0 by $-x_t, -x_0$

In physical terms, the probability that the Brownian particle wanders a certain distance in a given interval of time is INDEPENDENT

i) of the POSITION from which it STARTS to wander;

ii) of the TIME when it STARTS to wander;

iii) of the DIRECTION in which it wanders.

- Higher-dimensional (d) generalization

$$\vec{x} = (x_1, x_2, \dots, x_d) \in R^d$$

- diffusion eq.:

$$\frac{\partial W(\vec{x}_t, t \mid \vec{x}_0, t_0)}{\partial t} = D \, \Delta W(\vec{x}_t, t \mid \vec{x}_0, t_0) \quad , t > t_0$$

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_d^2}$$ – Laplacian operator

- solution:

$$W(\vec{x}_t, t \mid \vec{x}_0, t_0) = \frac{1}{[4\pi D(t-t_0)]^{d/2}} \, e^{-\frac{(\vec{x}_t - \vec{x}_0)^2}{4D(t-t_0)}}$$

- normalization:

$$\int_{-\infty}^{\infty} d^d x_t \; W(\vec{x}_t, t \mid \vec{x}_0, t_0) = 1$$

<u>Semigroup property of the transition probability.</u>

- Consider three instants of time:

$$t_0 < t' < t$$

and the corresponding probability densities:

$$w(x_0, t_0) \; , \; w(x', t') \; , \; w(x, t)$$

$$w(x,t) = \int_{-\infty}^{\infty} dx' \; W(x,t \mid x',t') \, w(x',t')$$

$$w(x',t') = \int_{-\infty}^{\infty} dx_0 \; W(x',t' \mid x_0,t_0) \, w(x_0,t_0)$$

$$\Rightarrow w(x,t) = \int_{-\infty}^{\infty} dx_0 \, dx' \; W(x,t \mid x',t') \, W(x',t' \mid x_0,t_0) \, w(x_0,t_0) \quad (19)$$

But we have also

$$w(x,t) = \int_{-\infty}^{\infty} dx_0 \; W(x,t \mid x_0,t_0) \, w(x_0,t_0) \quad (20)$$

Putting together (19) and (20) we obtain

$$\boxed{W(x,t \mid x_0,t_0) = \int_{-\infty}^{\infty} dx' \; W(x,t \mid x',t') \, W(x',t' \mid x_0,t_0), \quad t_0 < t' < t} \quad (21)$$

Einstein-Smoluchowski-Kolmogorov-Chapman (ESKC) relation

- ESKC relation expresses the SEMIGROUP PROPERTY of the transition probability $W(x,t|x_0,t_0) \rightarrow$ Markov process!

- Semigroup property for a general set of objects $F(t)$:

  $(\exists)$ $*$ - composition law, s.t.

$$F(t) * F(t') = F(t+t') \qquad (22)$$

- in the case of $W(x,t|x_0,t_0) \equiv W(x-x_0, t-t_0)$, the composition law is the integration over the intermediate coordinate

$$W(x-x_0, t-t_0) = \int_{-\infty}^{\infty} dx' \, W(x-x', t-t') \, W(x'-x_0, t'-t_0) \qquad (23)$$

$$\parallel \qquad\qquad\qquad\qquad\qquad\qquad \parallel$$

$$W(t-t_0) = W((t-t')+(t'-t_0)) = W(t-t') * W(t'-t_0)$$

- Why "semigroup" and not "group" property?

  Because the inverse element cannot be defined.

  (An inverse element would correspond to motion backward in time, $W(x,t|x_0,t_0)$, $t < t_0$, in which case

$$W(x,t|x_0,t_0) = \frac{1}{\sqrt{4\pi D(t-t_0)}} \, e^{-\frac{(x-x_0)^2}{4D(t-t_0)}}, \quad t-t_0 < 0$$

  has exponential GROWTH!

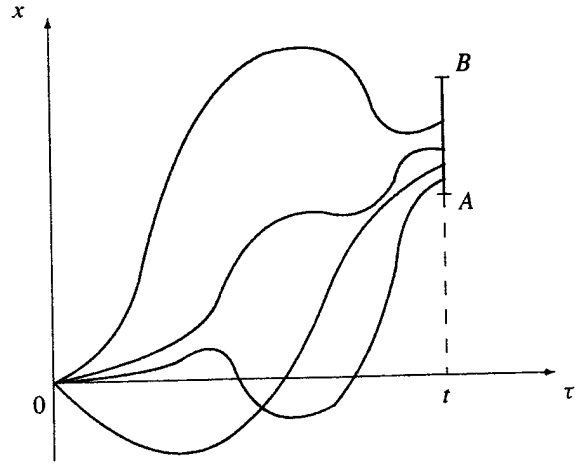
- the semigroup property is specific for Markow processes, i.e. random motion without memory and homogeneous in time.

Wiener path integrals

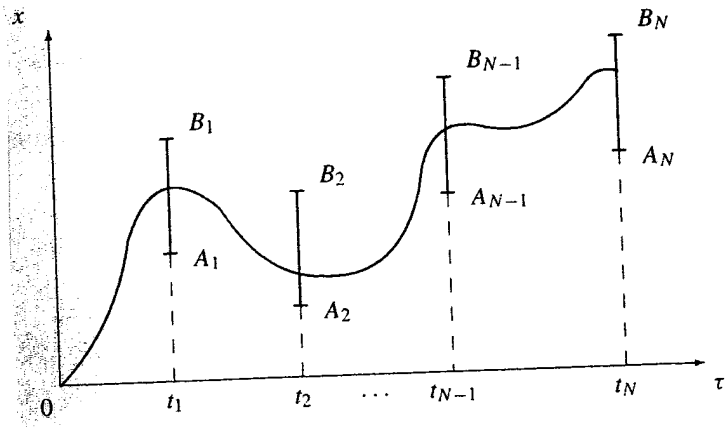
- Markovian property of the Brownian motion:

– the probability to find the Brownian particle after the time t ANYWHERE in the piece AB:

$$\int_A^B dx\, w(x,t)$$



– the probability that the Brownian particle passes through the gates $A_1B_1, A_2B_2, \ldots, A_NB_N$ at the moments of time $t_1, t_2, \ldots, t_N$



- Markov property: the succesive displacements of the Brownian particle are STATISTICALLY INDEPENDENT $\Rightarrow$ the probability of the compound event is the product of the probabilities of the succesive events

$$\mathcal{P}\{x(t_1)\in A_1B_1,\ x(t_2)\in A_2B_2, \ldots, x(t_N)\in A_NB_N\} =$$

$$= \int_{A_1}^{B} dx_1 \frac{e^{-\frac{x^2}{4Dt_1}}}{\sqrt{4\pi D t_1}} \int_{A_2}^{B_2} dx_2 \frac{e^{-\frac{(x_2-x_1)^2}{4D(t_2-t_1)}}}{\sqrt{4\pi D(t_2-t_1)}} \cdots$$

$$\times \int_{A_N}^{B_N} dx_N \frac{e^{-\frac{(x_N-x_{N-1})^2}{4D(t_N-t_{N-1})}}}{\sqrt{4\pi D(t_N-t_{N-1})}} \qquad (24)$$

- Markov process : a stochastic process with independent increments (without memory). The corresponding transition probability is arbitrary, up to the normalization

$$W(x_t, t \mid x_0, t_0)\, dx_t = \mathcal{P}\{x(t) \in [x_t, x_t + dx_t], x(t_0) = x_0\}, \quad t_0 < t$$

- Wiener process : a stochastic process with initial distribution

$$w(x,t) \xrightarrow[t \to t_0]{} \delta(x - x_0)$$

and transition probability

$$W(x,t \mid x_0, t_0) = \frac{1}{\sqrt{4\pi D(t-t_0)}}\, e^{-\frac{(x-x_0)^2}{4D(t-t_0)}}$$

- <u>Probability</u> that the Brownian particle moves along a <u>TRAJECTORY $x(\tau)$</u>

  - transition to continuous time :

$$t_i - t_{i-1} \equiv \Delta t_i \to 0 \quad , \quad 1 \le i \le N$$

→ probability that the Brownian particle passes through an infinite number of infinitesimal gates dx along the trajectory $x(\tau)$:

$$\lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} e^{-\sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{4D(t_i - t_{i-1})}} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{4\pi D(t_i - t_{i-1})}}$$

$$= \lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} e^{-\frac{1}{4D} \sum_{i=1}^{N} \left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}}\right)^2 \Delta t_i} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{4\pi D\, \Delta t_i}}$$

$$\equiv \boxed{e^{-\frac{1}{4D} \int_0^t d\tau\, \dot{x}^2(\tau)} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{4\pi D\, d\tau}}} \qquad (25)$$

• Notations :

$C\{x_1, t_1 ; x_2, t_2\}$: set of trajectories starting at $x(t_1) = x_1$ and ending at $x(t_2) = x_2$

$C\{x_1, t_1; AB, t_2\}$: set of trajectories starting at $x(t_1) = x_1$ and ending in the gate AB at time t_2

$C\{x_1, t_1; t_2\}$: set of trajectories starting at $x(t_1) = x_1$ and ending at any point (arbitrary endpoint)

- Evaluate again the probability to find the Brownian particle anywhere in the gate AB at the moment t, i.e. SUM the probabilities (25) over the set $C\{0,0; AB, t\}$ of ALL TRAJECTORIES which end up in the interval AB at time t (continuous set!)

$$\mathcal{P}\{x(t) \in AB\} = \boxed{\int_{C\{0,0;AB,t\}} e^{-\frac{1}{4D}\int_0^t d\tau\, \dot{x}^2(\tau)} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{4\pi D d\tau}}}$$ Wiener PATH INTEGRAL

$$= \int_A^B w(x,t)\, dx = \int_A^B \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x^2}{4Dt}} \qquad (26)$$

- For $A = B = x_t$

$$W(x_t, t \mid 0,0) = \int_{C\{0,0;x_t,t\}} d_W x(\tau) = \frac{1}{\sqrt{4\pi D t}}\, e^{-\frac{x_t^2}{4Dt}} \qquad (27)$$

with WIENER MEASURE as integration measure :

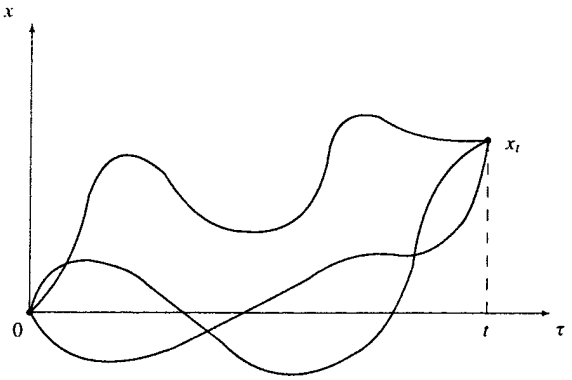
$$\boxed{d_W x(\tau) \equiv e^{-\frac{1}{4D}\int_0^t \dot{x}^2(\tau)\, d\tau} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{4\pi D d\tau}}} \qquad (28)$$

- UNCONDITIONAL (full, absolute) WIENER MEASURE : the integration is taken over the set of trajectories with ARBITRARY ENDPOINT $C\{x_1, t_1; t_2\}$

$$\int_{C\{x_1,t_1;t_2\}} d_W X(\tau) = 1 , \qquad (29)$$

since the probability that the Brownian particle ends up anywhere is equal to unity.

- CONDITIONAL WIENER MEASURE: the integration is taken over the set $C\{0,0;x_t,t\}$ of paths with FIXED end point.



Remark :

$$\int_{C\{0,0;t\}} d_W X(\tau) = \int_{-\infty}^{\infty} dx_t \int_{C\{0,0;x_t,t\}} d_W X(\tau) \qquad (30)$$

- ESKC relation in terms of Wiener measure and path integral

$$\int_{C\{x_0,0;x_t,t)} d_W X(\tau) = \int_{-\infty}^{\infty} dx' \int_{C\{x_0,0;x',t')} d_W X(\tau) \int_{C\{x',t';x_t,t)} d_W X(\tau) \qquad (31)$$

- Average of a functional $F[X(\tau)]$ with Wiener measure :

$$\int_C F[X(\tau)]\, d_W X(\tau) = \int_C F[X(\tau)]\, e^{-\frac{1}{4D}\int_0^t \dot{x}^2(\tau)d\tau} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{4\pi D d\tau}} \qquad (32)$$

- Set of trajectories contributing to Wiener's path integral

<u>Wiener's theorem</u> : The Wiener path integral is equal to zero over both the set of discontinuous and the set of differentiable trajectories

$\Rightarrow$ the relevant trajectories for the Wiener path integral are CONTINUOUS and NONDIFFERENTIABLE

- Some intuitive arguments
  - Evaluate the velocity of a Brownian particle

$$\overline{x^2} = \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \, w(x,t) \, dx = \int_{-\infty}^{\infty} x^2 \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi D t}} \, dx = \frac{t}{2} \quad (33)$$

$$\frac{\sqrt{\overline{x^2}}}{t} \sim \frac{\sqrt{t}}{t} \xrightarrow[t \to 0]{} \infty \quad (34)$$

(instantaneous speed at any moment of time)

$\Rightarrow$ nondifferentiability of the trajectories

fully "zig-zag"-like trajectories

Integration of functionals. Some simple examples

Example 1

$$F[x(\tau)] = x(t_\ell) = x_\ell \quad (34)$$

$$\overline{x_\ell} = \langle x_\ell \rangle = \int_{C\{x_0,0;t\}} x_\ell \, d_W x(\tau) = \int_{C\{x_0,0;t\}} x_\ell \, e^{-\int_0^t \dot{x}^2 d\tau} \prod_{\tau=0}^{t} \frac{dx(\tau)}{\sqrt{\pi d\tau}}, \quad (35)$$

where we considered $4D = 1$

- evaluate the integral by discrete approximation (DISCRETE-TIME or TIME-SLICED approximation)
  - divide the interval $[0,t]$ in N slices

$$\Delta t_i = t_i - t_{i-1} = \varepsilon = t/N \quad , \quad i = 1, \dots, N$$

$$\overline{x_\ell} = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}} x_\ell \prod_{i=1}^{N} \frac{dx_i}{\sqrt{\pi \Delta t_i}}$$

$$= \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{\ell-1} e^{-\sum_{i=1}^{\ell} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}} \prod_{i=1}^{\ell-1} \frac{dx_i}{\sqrt{\pi \Delta t_i}} \int_{-\infty}^{\infty} x_\ell \frac{dx_\ell}{\sqrt{\pi \Delta t_\ell}}$$

$$\times \underbrace{\underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{N-\ell} e^{-\sum_{i=\ell+1}^{N} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}} \prod_{i=\ell+1}^{N} \frac{dx_i}{\sqrt{\pi \Delta t_i}}}_{\int_{-\infty}^{\infty} dx_N \, W(x_N, t_N; x_\ell, t_\ell) = 1 \quad (\text{see } (18))} \quad (36)$$

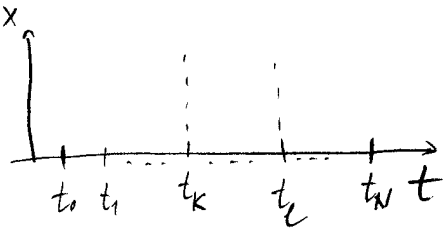
$$= \int_{-\infty}^{\infty} W(x_\ell, t_\ell; x_0, 0) \cdot x_\ell \frac{dx_\ell}{\sqrt{\pi \Delta t_\ell}} \cdot 1 = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x_\ell - x_0)^2}{t_\ell}}}{\sqrt{\pi t_\ell}} x_\ell \, dx_\ell = \underline{\underline{x_0}} \quad (37)$$

$\Rightarrow$ the average of x_ℓ is x_0, independent of t_ℓ, i.e. $\overline{x_\ell} = x_0$ ALWAYS - the point where the particle started from at the initial moment, in accord with the assumptions i), ii), iii).

Example 2

$$F[x(\tau)] = x(t_k)\,x(t_\ell) = x_k x_\ell\,, \quad 0 < t_k < t_\ell < t$$

$$\overline{x(t_k)x(t_\ell)} = \int_{C\{x_0,0;t\}} x(t_k)\,x(t_\ell)\,d_W x(\tau)$$



$$= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\sum_{i=1}^{N}\frac{(x_i-x_{i-1})^2}{t_i-t_{i-1}}}\, x_k x_\ell \prod_{i=1}^{N}\frac{dx_i}{\sqrt{\pi\Delta t_i}}$$

$$= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{N-\ell} e^{-\sum_{i=\ell+1}^{N}\frac{(x_i-x_{i-1})^2}{t_i-t_{i-1}}} \prod_{i=\ell+1}^{N}\frac{dx_i}{\sqrt{\pi\Delta t_i}} \times$$

$$\times \int_{-\infty}^{\infty} x_\ell \frac{dx_\ell}{\sqrt{\pi\Delta t_\ell}} \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{\ell-k-1} e^{-\sum_{i=k+1}^{\ell}\frac{(x_i-x_{i-1})^2}{t_i-t_{i-1}}} \prod_{i=k+1}^{\ell-1}\frac{dx_i}{\sqrt{\pi\Delta t_i}} \times$$

$$\times \int_{-\infty}^{\infty} x_k \frac{dx_k}{\sqrt{\pi\Delta t_k}} \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{k-1} e^{-\sum_{i=1}^{k}\frac{(x_i-x_{i-1})^2}{t_i-t_{i-1}}} \prod_{i=1}^{k}\frac{dx_i}{\sqrt{\pi\Delta t_i}}$$

$$= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_{-\infty}^{\infty} dx_N\, W(x_N,t_N;x_{N-\ell},t_{N-\ell}) \times$$

$$\times \int_{-\infty}^{\infty} x_\ell\, dx_\ell \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{i=k+1}^{\ell}\frac{e^{-\frac{(x_i-x_{i-1})^2}{\Delta t_i}}}{\sqrt{\pi\Delta t_i}} \prod_{i=k+1}^{\ell-1} dx_i$$

$$\times \int_{-\infty}^{\infty} x_k\, dx_k \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{i=1}^{k}\frac{e^{-\frac{(x_i-x_{i-1})^2}{\Delta t_i}}}{\sqrt{\pi\Delta t_i}} \prod_{i=1}^{k-1} dx_i$$

$$= 1\cdot \int_{-\infty}^{\infty} x_\ell\, dx_\ell \cdot W(x_\ell,t_\ell \mid x_k,t_k)\cdot \int_{-\infty}^{\infty} x_k\, dx_k\, W(x_k,t_k;x_0,0)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_\ell\, dx_k\; x_\ell x_k\, \frac{e^{-\frac{(x_\ell-x_k)^2}{t_\ell-t_k}}}{\sqrt{\pi(t_\ell-t_k)}}\, \frac{e^{-\frac{(x_k-x_0)^2}{t_k}}}{\sqrt{\pi t_k}}$$

- use also $\int x^2 e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$

$$\overline{x(t_k)\,x(t_\ell)} = \int_{-\infty}^{\infty} dx_k \, \frac{e^{-\frac{(x_k-x_0)^2}{t_k}}}{\sqrt{\pi t_k}} x_k \int_{-\infty}^{\infty} dx_\ell \, \frac{e^{-\frac{(x_\ell-x_k)^2}{t_\ell-t_k}}}{\sqrt{\pi(t_\ell-t_k)}} x_\ell$$

$$= \int_{-\infty}^{\infty} dx_k \, x_k^2 \, \frac{e^{-\frac{(x_k-x_0)^2}{t_k}}}{\sqrt{\pi t_k}} = \frac{t_k}{2}$$

In general,

$$\overline{x(t_k)\,x(t_\ell)} = \frac{\min(t_k, t_\ell)}{2}$$