Yang-Mills field

- For definitness, we consider SU(2) gauge group, with antihermitian generators:

$$[T^a, T^b] = g f^{abc} T^c \qquad (25)$$

$$\omega(x) = e^{\varepsilon(x)} = e^{\varepsilon^a(x) T^a} \in SU(2) \qquad (26)$$

- Lagrangian of Yang-Mills fields:

$$\mathcal{L}_{YM} = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$$

$$= -\frac{1}{2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \frac{g}{2}\varepsilon_{abc}\left(A^b_\mu A^c_\nu - A^b_\nu A^c_\mu\right) - \frac{1}{2}F^a_{\mu\nu}\right)F^{a\mu\nu}$$

$$= E_i^a \dot{A}_i^a - \mathcal{H}(E_i^a, A_i^a) + A_0^a\left(\partial_i E_i^a - g f_{abc} A_i^b E_i^c\right) \qquad (27)$$

with $\mathcal{H}(E_i^a, A_i^a) = \frac{1}{2}\left[(E_i^a)^2 + (G_i^a)^2\right]$, (28)

$$E_i^a = F_{i0}^a, \quad G_i^a = \frac{1}{2}\varepsilon_{lji} F^a_{jl}$$

- By the same analysis as for the electromagn. field:

$A_i^a(x)$ – generalized coordinate

$E_i^a(x)$ – canonically conjugated momentum

$$\{E_i^a(x), A_j^b(y)\} = \delta_{ij}\,\delta_{ab}\,\delta(x-y) \qquad (29)$$

$$\{E_i^a(x), E_j^b(y)\} = \{A_i^a(x), A_j^b(y)\} = 0 \qquad (30)$$

$A_0^a(x)$ – Lagrange multiplier

$$C^a(x) = \partial_i E_i^a(x) - g f_{abc} A_i^b E_i^c = 0$$ PRIMARY constraint,

obeying $\{C^a(x), C^b(y)\} = g\,\varepsilon_{abd}\, C^d(x)\,\delta(x-y) \sim 0$

$\{\int d^3x\, \mathcal{H}(x), C^a(y)\} = 0 \Rightarrow$ no more constraints!

- GAUGE FIXING - subsidiary constraint :

$$\partial_i A_i^a(x) = 0 \,, \qquad (31) \qquad \text{Coulomb-type}$$

fulfilling $\{\partial_i A_i^a(x), \partial_j A_j^b(y)\} = 0$

$$\{c^a(x), \partial_i A_i^b(y)\} = M_{YC}(A_i^a)\,\delta(x-y), \qquad (32)$$

where $M_{YC}^{ab}(A_i^a) = \Delta\,\delta_{ab} + g\,\varepsilon_{abc} A_i^c \partial_i \qquad (33)$

- Remark that

i) $\det|M_{YC}^{ab}(A)| \neq 0 \qquad (34)$.

ii) $M_{YC}^{ab}(A)$ depends on the gauge fields !

- write the path integral for YM fields in Coulomb gauge:

$$\langle 1 \rangle = \int \prod_{x,a} \mathcal{D}A_i^a(x)\,\mathcal{D}E_i^a(x)\,\mathcal{D}A_0^a(x)\,\det|M_{YC}^{ab}(A)|$$
$$\times\,\delta(\partial_i A_i^a(x))\,e^{i\int d^4x\left[E_k^a \dot{A}_k^a - \mathcal{H}(x) + A_0^a(x)\,C^a(x)\right]} \qquad (35)$$

$$= \int \prod_{x,a} \mathcal{D}A_\mu^a(x)\,\det|M_{YC}^{ab}(A)|\,\delta(\partial_i A_i^a(x))\,e^{i\int d^4x\,\mathcal{L}_{YM}} \qquad (36)$$

- expression (36), which is not Lorentz covariant due to the Coulomb gauge condition can be made covariant by the FADDEEV - POPOV trick (Coulomb gauge $\rightarrow$ Lorentz gauge)

- Consider the matrix $\Delta_c(A_\mu)$, defined by:

$$\Delta_c(A_\mu)\Big|_{\partial_i A_i = 0} \int \prod_{a,x} \mathcal{D}\omega(x)\, \delta(\partial_i A_i^\omega(x)) = 1 \qquad (37)$$

- the integration measure $\mathcal{D}\omega(x)$ is invariant with respect to the gauge group

$\Downarrow$

$\Delta_c(A_\mu)$ invariant under gauge transf,

ie. $\Delta_c(A_\mu^\omega) = \Delta_c(A_\mu)$ (38)

- since $\partial_i A_i^a(x) = 0$ (Coulomb gauge), it follows that we get contribution from δ-funct. in (37) for $\omega = 1$, therefore we can integrate in a neighbourhood of unity, i.e. for infinitesimal gauge transf:

$$\omega(x) = 1 + \varepsilon(x) \qquad (39)$$

$$A_\mu^\omega(x) = \omega(x) A_\mu(x) \omega^{-1}(x) + \omega^{-1}(x)\partial_\mu \omega(x)$$

$$= (1+\varepsilon(x)) A_\mu(x) (1-\varepsilon(x)) + (1-\varepsilon(x))\partial_\mu(1+\varepsilon(x))$$

$$= A_\mu(x) - [A_\mu(x), \varepsilon(x)] + \partial_\mu \varepsilon(x) \qquad (40)$$

$\Downarrow$

$$\partial_i A_i^\omega(x) = \underbrace{\partial_i A_i(x)}_{=0} - [\underbrace{\partial_i A_i(x)}_{=0}, \varepsilon(x)] - [A_i(x), \partial_i \varepsilon(x)] + \partial_i \partial_i \varepsilon(x)$$

$$= \Delta \varepsilon(x) - [A_i(x), \partial_i \varepsilon(x)]$$

$$= \Delta \varepsilon(x) - A_i^a(x)\partial_i \varepsilon^b(x) \underbrace{[T^a, T^b]}_{g f^{abc} T^c}$$

$$= (\Delta \delta_{ab} + g f_{abc} A_i^c \partial_i)\varepsilon(x) \underset{(33)}{=} M_{Yc}(A_\mu)\varepsilon(x) \qquad (41)$$

- then (37) becomes

$$\Delta_c(A_\mu) \underbrace{\int \delta\left(M_{Yc}\,\varepsilon(x)\right) \underset{\substack{\parallel \\ 1}}{J} \prod_x \mathcal{D}\varepsilon(x)}_{\frac{1}{\det|M_{Yc}(A_\mu)|}} = 1 \qquad (42)$$

$$\Rightarrow \qquad \Delta_c(A_\mu)\Big|_{\partial_i A_i(x)=0} = \det\left|M_{Yc}(A_\mu)\right| \qquad (43)$$

- in the same manner, one proves that

$$\Delta_L(A_\mu)\Big|_{\partial^\mu A_\mu(x)=0} = \det\left|M_{YL}(A_\mu)\right|, \qquad (44)$$

where

$$M^{ab}_{YL}(A_\mu) = \Box\,\delta_{ab} + g\, f_{abc}\, A^c_\mu(x)\,\partial_\mu \qquad (45)$$

- in order to go from the Coulomb to the Lorentz gauge in (36), we shall INSERT THE UNITY

$$\Delta_L(A_\mu) \int \mathcal{D}\omega(x)\, \delta\left(\partial_\mu A^\omega_\mu(x)\right) = 1 \qquad (46)$$

- thus, (36) becomes

$$\langle 1 \rangle = \int \prod_{a,x} \mathcal{D}\omega(x)\, \mathcal{D}A^a_\mu(x)\, e^{i\int d^4x\, \mathcal{L}_{YM}}\, \delta(\partial_i A^a_i(x))$$
$$\times \det\left|M_{Yc}(A_\mu)\right|\, \Delta_L(A_\mu)\, \delta\left(\partial_\mu A^\omega_\mu(x)\right) \qquad (47)$$

- now we re-arrange the factors so as to get the unity (37), and $A_\mu \to A^{\omega^{-1}}_\mu$

$$\langle 1 \rangle = \int \prod_{a,x} \mathcal{D}A^a_\mu(x)\, \mathcal{D}\omega(x)\, \delta\left(\partial_i A^{\omega^{-1}}_i(x)\right) \Delta_c(A_\mu)\, \Delta_L(A_\mu)\, \delta\left(\partial_\mu A_\mu(x)\right)$$
$$\times\, e^{i\int d^4x\, \mathcal{L}_{YM}}$$

$$= \int \prod_{a,x} \mathcal{D}A^a_\mu(x)\, \det\left|M_{YL}(A_\mu)\right|\, \delta\left(\partial^\mu A_\mu(x)\right)\, e^{i\int d^4x\, \mathcal{L}_{YM}}$$

- the new expression, obtained using the gauge invar. of $\Delta_c(A_\mu)$, $\Delta_L(A_\mu)$ and $\mathcal{D}A_\mu(x)$, is Lorentz covariant:

$$\boxed{\langle 1 \rangle = \int \prod_{x,a} \mathcal{D}A^a_\mu(x) \, \det|M_{YL}(A_\mu)| \, \delta(\partial^\mu A_\mu(x)) \, e^{i\int d^4x \, \mathcal{L}_{YM}}} \quad (48)$$

- Remark that the determinant of $M_{YL}(A_\mu)$ depends on the gauge fields. It can be rewritten as a path integral, by using the integral representation of a determinant as Gaussian integral over Grassmann variables

$$\det|M_{YL}(A_\mu)| = \int \prod_{a,x} \mathcal{D}\bar{c}^a(x) \, \mathcal{D}c^a(x) \, e^{i\int d^4x \, \bar{c}^a(x)(\Box\delta_{ab} + g f_{abc} A^c_\mu \partial_\mu) c^b(x)} \quad (49)$$

$$= \int \prod_{a,x} \mathcal{D}\bar{c}^a(x) \mathcal{D}c^a(x) \, e^{i\int d^4x \, \bar{c}^a(x)(\partial^\mu \partial_\mu c - g[A_\mu, c])^a} \quad (50)$$

- The fields $\bar{c}^a(x)$, $c^a(x)$ are SCALAR in the sense of Lorentz representation, but satisfy fermionic statistics due to their Grassmannian character. They are called FADDEEV-POPOV GHOSTS and are, obviously, fictitious (nonphysical) degrees of freedom. Thus, they appear in intermediate calculations, but the observables do NOT depend on them.

- These "particles" were predicted in a way by Feynman (1963), who observed that YM theory in which the determinant is ignored leads to unitary results only if loops of scalar particles, but with $\underbrace{\text{MINUS}}_{\text{fermionic!}}$ sign, are subtracted consistently.

The mathematical solution was given by Faddeev and Popov (1967).

General relativistically-invariant gauges:

$$\partial_\mu A^{b\mu}(x) = a^b(x) \qquad (51)$$

generalized Lorenz gauge,

where $a^b(x)$ is an arbitrary function.

$$\Delta_a(A_\mu) \int \prod_{x,a} \mathcal{D}\omega(x)\, \delta(\partial^\mu A^{b\omega}_\mu - a^b(x)) = 1 \qquad (52)$$

$$\Downarrow$$

$$\Delta_a(A_\mu)\Big|_{\partial^\mu A^b_\mu = a^b} = \det|M_a| \,, \qquad (53)$$

where $M_a^{bc} = \Box \delta_{bc} + g\, \varepsilon_{bcd} (A^d_\mu \partial_\mu + \partial_\mu A^d_\mu)$ (54)

Remark: M_a does not depend on $a^b(x)$!

$$\langle 1 \rangle = \int \prod_{x,b} \mathcal{D}A^b_\mu(x)\, \delta(\partial^\mu A^b_\mu(x) - a^b(x)) \det|M_a|\, e^{-i\int \mathcal{L}_{YM} d^4x} \qquad (55)$$

- Due to gauge-invariance, the transition amplitude does not depend on $a^b(x)$, therefore we can integrate over these functions, with a weight like

$$e^{-i\frac{1}{2\alpha}\int (a^b(x))^2 d^4x} \qquad (56).$$

- then (55) becomes

$$\langle 1 \rangle = \int \prod_{x,b} \mathcal{D}A^b_\mu(x) \det|M_a|\, e^{i\int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2\right)} \qquad (57)$$

α-gauge

(observables should not depend on α due to gauge invariance!)

Remarks:

i) In non-Abelian gauge theories one can choose gauges such that no ghosts appear. Such a gauge is the AXIAL GAUGE

$$n_\mu A^\mu(x) = 0 . \qquad (58)$$

(The price to pay is a much more complicated propagator for the YM fields)

ii) In QED, if one chooses a nonlinear gauge, for example, the determinant will depend on the gauge fields and ghosts will appear.