- use also $\int x^2 e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$

$$\overline{x(t_k)\,x(t_\ell)} = \int_{-\infty}^{\infty} dx_k \, \frac{e^{-\frac{(x_k-x_0)^2}{t_k}}}{\sqrt{\pi t_k}}\, x_k \int_{-\infty}^{\infty} dx_\ell \, \frac{e^{-\frac{(x_\ell-x_k)^2}{t_\ell-t_k}}}{\sqrt{\pi(t_\ell-t_k)}}\, x_\ell$$

$$= \int_{-\infty}^{\infty} dx_k \, x_k^2 \, \frac{e^{-\frac{(x_k-x_0)^2}{t_k}}}{\sqrt{\pi t_k}} = \frac{t_k}{2}$$

In general,

$$\overline{x(t_k)\,x(t_\ell)} = \frac{\min(t_k, t_\ell)}{2}$$

Example 3

$$F[x(\tau)] = F\left[\int_0^t a(\tau)\,x(\tau)\,d\tau\right] \qquad , \; 4D = 1$$

$$I = \int_{C\{0,0;\,t\}} F\left[\int_0^t a(\tau)\,x(\tau)\,d\tau\right] d_W x(\tau)$$

$$= \lim_{\substack{\Delta t_i \to 0 \\ N\to\infty}} I_N = \lim_{\substack{\Delta t_i \to 0 \\ N\to\infty}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F\left[\sum_{i=1}^{N} a_i\,x_i\,\Delta t_i\right] e^{-\sum_{i=1}^{N}\frac{(x_i-x_{i-1})^2}{\Delta t_i}} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{\pi \Delta t_i}},$$

where

$$a_i = a(t_i)$$
$$x_i = x(t_i)$$

Define

$$\begin{cases} x_1 = y_1 \\ x_2 - x_1 = y_2 \\ \vdots \\ x_N - x_{N-1} = y_N \end{cases} \Rightarrow \begin{cases} y_1 = x_1 \\ y_1 + y_2 = x_2 \\ \vdots \\ y_1 + y_2 + \dots + y_{N-1} = x_{N-1} \\ y_1 + y_2 + \dots + y_N = x_N \end{cases} \qquad J = 1$$

(Jacobian of the transformation)

$$\sum_{i=1}^{N} a_i x_i \Delta t_i = a_1 x_1 \Delta t_1 + a_2 x_2 \Delta t_2 + \ldots + a_N x_N \Delta t_N$$

$$= a_1 y_1 \Delta t_1 + a_2 (y_1+y_2) \Delta t_2 + \ldots + a_N (y_1+y_2+\ldots+y_N) \Delta t_N$$

$$= y_1 (a_1 \Delta t_1 + a_2 \Delta t_2 + \ldots + a_N \Delta t_N) + y_2 (a_2 \Delta t_2 + \ldots + a_N \Delta t_N) + \ldots + y_N a_N \Delta t_N$$

$$= y_1 \underbrace{\sum_{i=1}^{N} a_i \Delta t_i}_{A_1} + y_2 \underbrace{\sum_{i=2}^{N} a_i \Delta t_i}_{A_2} + \ldots y_N \underbrace{a_N \Delta t_N}_{A_N}$$

$$\left(A_i = \sum_{k=i}^{N} a_k \Delta t_k \right)$$

$$= \underbrace{y_1 A_1}_{z_1} + \underbrace{y_2 A_2}_{z_2} + \ldots + \underbrace{y_N A_N}_{z_N}$$

Then

$$I_N = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} F(y_1 A_1 + y_2 A_2 + \ldots + y_N A_N)\, e^{-\sum_{i=1}^{N} \frac{y_i^2}{\Delta t_i}} \cdot 1 \prod_{i=1}^{N} \frac{dy_i}{\sqrt{\pi \Delta t_i}}$$

J

$$= \underbrace{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}}_{N} F(z_1+z_2+\ldots+z_N)\, e^{-\sum_{i=1}^{N} \frac{z_i^2}{A_i^2 \Delta t_i}} \prod_{i=1}^{N} \frac{dz_i}{\sqrt{\pi A_i^2 \Delta t_i}}$$

Change of variables

$$z_1 + z_2 = \xi$$
$$z_2 = \eta$$

(Remark: $F(z_1+z_2+\ldots+z_N) = F(\xi+z_3+\ldots+z_N)$, i.e. F does NOT depend on η!)

and integrate over η:

$$\int_{-\infty}^{\infty} d\eta\, \frac{e^{-\frac{(\xi-\eta)^2}{A_1^2 \Delta t_1}}}{\sqrt{\pi A_1^2 \Delta t_1}} \cdot \frac{e^{-\frac{\eta^2}{A_2^2 \Delta t_2}}}{\sqrt{\pi A_2^2 \Delta t_2}} = \frac{e^{-\frac{\xi^2}{A_1^2 \Delta t_1 + A_2^2 \Delta t_2}}}{\sqrt{\pi (A_1 \Delta t_1 + A_2 \Delta t_2)}}$$

(use semigroup property!)

Then

$$I_N = \underbrace{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}}_{N-1} F(\xi + z_3 + \ldots + z_N)\, e^{-\frac{\xi^2}{A_1^2 \Delta t_1 + A_2^2 \Delta t_2}}\, e^{-\sum_{i=3}^{N} \frac{z_i^2}{A_i^2 \Delta t_i}} \frac{d\xi}{\sqrt{\pi (A_1^2 \Delta t_1 + A_2^2 \Delta t_2)}} \prod_{i=3}^{N} \frac{dz_i}{\sqrt{\pi A_i^2 \Delta t_i}}$$

Continue the procedure like above

$$\xi + z_3 \to \xi$$
$$z_3 \to \eta$$

, integrate over η etc.

Finally we get:

$$I_N = \int_{-\infty}^{\infty} F(\xi)\, e^{-\frac{\xi^2}{\sum_{i=1}^{N} A_i^2 \Delta t_i}} \frac{d\xi}{\sqrt{\pi \sum_{i=1}^{N} A_i^2 \Delta t_i}}$$

$$I = \lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} I_N = \int_{-\infty}^{\infty} F(\xi)\, e^{-\frac{\xi^2}{\int_0^t A^2(\tau)d\tau}} \frac{d\xi}{\sqrt{\pi \int_0^t A^2(\tau) d\tau}},$$

where $A(\tau) = \int_\tau^t a(s)\,ds$, since $A_i = \sum_{k=i}^{N} a_i \Delta t_i$.

The conditions that $a(\tau)$ and F have to satisfy:

- $a(\tau)$ integrable on the interval $[0, t]$
- F continuous, s.t. the integral above exists

(F may even grow at infinity, since the exponential $e^{-\xi^2}$ provides damping.)

Example 4

$$F[x(\tau)] = e^{\lambda \int_0^t p(\tau)\, x^2(\tau)\, d\tau}$$

$$I = \int_{C\{0,0;t\}} e^{\lambda \int_0^t p(\tau) x^2(\tau) d\tau}\, d_W x(\tau) = \lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} I_N$$

$$= \lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{N} e^{\lambda \sum_{i=1}^{N} p_i x_i^2 \Delta t_i - \sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{\Delta t_i}} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{\pi \Delta t_i}}$$

$$= \lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{N} e^{-\sum_{i,j=1}^{N} a_{ij} x_i x_j} \frac{dx_1}{\sqrt{\pi \Delta t_1}} \cdots \frac{dx_N}{\sqrt{\pi \Delta t_N}},$$

where a_{ij} will be explicitly determined later.

Assume equal time slices and denote

$$\Delta t_1 = \Delta t_2 = \dots \Delta t_N \equiv \varepsilon$$

Gel'fand - Yaglom method

- Consider the integral

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^{N}dx_i \quad e^{-\sum_{i,j=1}^{N} a_{ij}x_ix_j}$$

- Change of variables, to diagonalize the exponent

$$y = \mathbb{O}\,x \quad , \quad \mathbb{O} - \text{rotation matrix} \Rightarrow \det\mathbb{O} = 1$$

$$\sum_{i,j} a_{ij}x_ix_j = \sum_{i,j} a_{ij}\sum_k \mathbb{O}_{ik}\,y_k\sum_\ell \mathbb{O}_{j\ell}\,y_\ell = \sum_{k,\ell}\underbrace{\sum_{i,j}(\mathbb{O}^T)_{ki}\,a_{ij}\,\mathbb{O}_{j\ell}}_{b_{k\ell}}\,y_k y_\ell$$

$$= \sum_{k,\ell} b_{k\ell}\,y_k y_\ell$$

$$B = \mathbb{O}^T A\,\mathbb{O} \quad , \quad A = (a_{ij}) - N\times N \text{ matrix}$$
$$B = (b_{k\ell}) - N\times N \text{ DIAGONAL matrix}$$

Then

$$\sum_{i,j} a_{ij}x_ix_j = \sum_k b_k y_k^2 ,$$

i.e.

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\sum_{i,j=1}^{N} a_{ij}x_ix_j}\prod_{i=1}^{N}dx_i = \int\cdots\int e^{-\sum_{k=1}^{N} b_k y_k^2}\cdot \underset{\nearrow}{1}\cdot\prod_{i=1}^{N}dy_k$$

Jacobian ($\det\mathbb{O}=1$)

$$= \int e^{-b_1y_1^2}dy_1\int e^{-b_2y_2^2}dy_2\cdots\int e^{-b_Ny_N^2}dy_N = \sqrt{\frac{\pi}{b_1}}\sqrt{\frac{\pi}{b_2}}\cdots\sqrt{\frac{\pi}{b_N}}$$

$$= \sqrt{\frac{\pi^N}{\prod_{k=1}^{N} b_k}} \underset{B\text{-diag!}}{=} \sqrt{\frac{\pi^N}{\det B}} = \sqrt{\frac{\pi^N}{\det A}} \;,\; \text{since } \det B = \underset{1}{\det\mathbb{O}^T}\det A\underset{1}{\det\mathbb{O}}$$

Using this result in our integral,

$$I_N = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\sum_{i,j=1}^{N} a_{ij}x_ix_j}\prod_{i=1}^{N}\frac{dx_i}{\sqrt{\pi\varepsilon}} =$$

$$= \frac{1}{(\sqrt{\pi\varepsilon})^N}\sqrt{\frac{\pi^N}{\det A}} = \frac{1}{\sqrt{\varepsilon^N\det A}} = \frac{1}{\sqrt{\det(\varepsilon A)}}$$

It remains to find the elements of the matrix A and in the end take the limit $\varepsilon \to 0$, $N \to \infty$.

$$\lambda \sum_{i=1}^{N} p_i x_i^2 \varepsilon - \sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{\varepsilon} = -\sum_{i,j=1}^{N} a_{ij} x_i x_j$$

- consider all the terms that contain x_i in the l.h.s.:

$$\lambda p_i x_i^2 \varepsilon - \frac{(x_i - x_{i-1})^2}{\varepsilon} - \frac{(x_{i+1} - x_i)^2}{\varepsilon} =$$

$$= \left(\lambda p_i \varepsilon - \frac{1}{\varepsilon} - \frac{1}{\varepsilon}\right) x_i^2 + \frac{2}{\varepsilon} x_i x_{i-1} + \frac{2}{\varepsilon} x_{i+1} x_i - \frac{1}{\varepsilon} x_{i-1}^2 - \frac{1}{\varepsilon} x_{i+1}^2$$

- we conclude

$$a_{ii} = \frac{2}{\varepsilon} - \lambda p_i \varepsilon \quad , \; i = \overline{1, N-1}$$

$$a_{i,i-1} = a_{i-1,i} = -\frac{1}{\varepsilon}$$

$$a_{i+1,i} = a_{i,i+1} = -\frac{1}{\varepsilon}$$

$$a_{NN} = \frac{1}{\varepsilon} - \lambda p_N \varepsilon$$

$$A = \begin{pmatrix} a_{11} & -\frac{1}{\varepsilon} & 0 & 0 & \cdots & 0 \\ -\frac{1}{\varepsilon} & a_{22} & -\frac{1}{\varepsilon} & 0 & \cdots & 0 \\ 0 & -\frac{1}{\varepsilon} & a_{33} & -\frac{1}{\varepsilon} & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & & 0 & -\frac{1}{\varepsilon} & a_N \end{pmatrix}$$

$\det \varepsilon A = ?$

Introduce the determinants

$$D_k^{(N)} = \det(\varepsilon A)_{k\ldots N} = \begin{vmatrix} \varepsilon a_k & -1 & 0 & \cdots & 0 \\ -1 & \varepsilon a_{k+1} & -1 & \cdots & 0 \\ 0 & & & & \\ \vdots & & & & \\ 0 & \cdots & 0 & -1 & \varepsilon a_N \end{vmatrix}$$

$$D_k^{(N)} = \varepsilon a_k D_{k+1}^{(N)} + (-1)(-1)^{1+2} \begin{vmatrix} -1 & -1 & 0 & \cdots & 0 \\ 0 & \varepsilon a_{k+2} & -1 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & -1 & \varepsilon a_N \end{vmatrix}$$

$$= \varepsilon a_k D_{k+1}^{(N)} + (-1) D_{k+2}^{(N)}$$

$$D_k^{(N)} = (2 - \lambda p_k \varepsilon^2) D_{k+1}^{(N)} - D_{k+2}^{(N)}$$

$$\Downarrow$$

$$\frac{D_k^{(N)} - 2 D_{k+1}^{(N)} + D_{k+2}^{(N)}}{\varepsilon^2} = -\lambda p_k D_{k+1}^{(N)} \qquad (*)$$

This is the approximation to the second derivative:
- introduce the variable

$$s = \frac{k-1}{N}$$

$N \to \infty$, $D_k^{(N)} \to D(s)$ (k is a label which becomes from discrete, continuous)

Then $(*)$ becomes

$$\boxed{\frac{d^2 D(\tau)}{d\tau^2} = -\lambda p(\tau) D(\tau)} \qquad (**)$$

$$\det(\varepsilon A) = \lim_{N \to \infty} D_1^{(N)} = D(0)$$

To find $D(0)$, we have to impose initial conditions to the differential equation $(**)$.

- Note that

$$D(t) = \lim_{N \to \infty} D_N^{(N)} = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \varepsilon a_N = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} (1 - \lambda p_N \varepsilon^2) = 1$$

- Also

$$\left.\frac{dD(\tau)}{d\tau}\right|_{\tau = t} = \lim_{\substack{N \to \infty \\ \varepsilon \to 0}} \frac{D_N^{(N)} - D_{N-1}^{(N)}}{\varepsilon}$$

$$D_N^{(N)} = \varepsilon a_N = 1 - \lambda p_N \varepsilon^2$$

$$D_{N-1}^{(N)} = \begin{vmatrix} 2-\lambda p_{N-1}\varepsilon^2 & -1 \\ -1 & 1-\lambda p_N \varepsilon^2 \end{vmatrix} = 2+\lambda^2 p_N p_{N-1}\varepsilon^4 - 2\lambda p_N \varepsilon^2 - \lambda p_{N-1}\varepsilon^2 - 1$$

$$\left.\frac{dD(\tau)}{d\tau}\right|_{\tau=t} = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \frac{1}{\varepsilon}\left(1-\lambda p_N\varepsilon^2 - p_N p_{N-1}\varepsilon^4 + 2\lambda p_N \varepsilon^2 + \lambda p_{N-1}\varepsilon^2 - 1\right)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\,\mathcal{O}(\varepsilon^2) = 0$$

Thus, the solution of the path integral

$$I = \int_{C\{0,0;t\}} d_W x(\tau)\; e^{\lambda \int_0^t p(\tau)x^2(\tau)\,d\tau}$$

is
$$I = \frac{1}{\sqrt{D(0)}},$$ where $D(0)$ is given

by the differential equation

$$\begin{cases} \dfrac{d^2 D(\tau)}{d\tau^2} = -\lambda p(\tau)\, D(\tau) \\ D(t) = 1 \\ \left.\dfrac{dD(\tau)}{d\tau}\right|_{\tau=t} = 0 \end{cases}$$

obtained by GEL'FAND - YAGLOM method