Quantization of systems with constraints

- Recall Hamiltonian formalism

$$L(q_i, \dot{q}_i, t) \longrightarrow H(p_i, q_i, t) \qquad (1)$$

$$(q_i, \dot{q}_i, t) \longrightarrow (p_i, q_i, t)$$

– p and q are INDEPENDENT variables, not like $q, \dot{q}$ which are dependent through $\dot{q} \equiv \frac{dq}{dt}$.

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{– canonically conjugated momentum to } q_i \qquad (2)$$

$$H(p_i, q_i, t) = \sum_{i=1}^{N} p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \qquad (3)$$

– dynamics given by the Hamilton eqs.
(the equiv. of Euler-Lagrange eq. in Lagrangian formalism)

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases} \qquad (4)$$

– Quantum mechanics of a system with finite degrees of freedom (d.o.f) goes in the limit to quantum field theory with infinite no. of d.o.f.

– Gauge field theories are the analogue, in the limit of mechanical systems with constraints.

– We shall prepare for the quantization of gauge field theories by first discussing the quantization by functional integral of mechanical systems with constraints (finite no. of d.o.f.!)

- Consider a mechanical system with n d.o.f

$$L(q_i, \dot{q}_i) \quad , \quad i = 1, \dots, n \qquad (5)$$

and m' constraints

$$\varphi_a(p_i, q_i) = 0 \qquad (6)$$ PRIMARY constraints

– use the method of Lagrange multipliers : include the constraints into the Hamiltonian by

$$H'(p_i q_i) = H(p_i q_i) + \sum_{a=1}^{m'} \lambda_a \varphi_a(p_i q_i) \, , \qquad (7)$$

where λ_a are arbitrary constants (Lagrange multipliers)

– the canonical eq. of motion (4), with the new Hamiltonian (7), become :

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} + \displaystyle\sum_{a=1}^{m'} \lambda_a \dfrac{\partial \varphi_a}{\partial p_i} = \{H', q_i\} \\ \dot{p}_i = - \dfrac{\partial H}{\partial q_i} - \displaystyle\sum_{a=1}^{m'} \lambda_a \dfrac{\partial \varphi_a}{\partial q_i} = \{H', p_i\} \end{cases} \qquad (8)$$

where $$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) \qquad (9)$$ Poisson bracket

– the primary constraints (6) have to be consistent with the eqs. of motion, i.e. they have to be CONSTANT in time :

$$\dot{\varphi}_a(p_i q_i) = 0 \, ,$$

$$\dot{\varphi}_a(p_i q_i) = \{H', \varphi_a\} = \{H, \varphi_a\} + \sum_{b=1}^{m'} \{\varphi_b, \varphi_a\} = 0 \qquad (10)$$

- if rel. (10) are not automatically fullfilled, then

$$\dot{\varphi}_a = \varphi_a^{(2)}(p_i, q_i) = 0 \qquad (11)$$ SECONDARY constraints

- the procedure continues until all the obtained constraints are constant in time (primary, secondary, ternary, ...)
- the final set of constraints is consistent with the eqs. of motion and all the constraints of the set are treated on the same footing.

• Assume that there are m constraints altoghether

$$\varphi_a(p_i q_i) = 0 \ , \quad a = 1,\dots m \qquad (12)$$

and the Hamiltonian of the constrained system becomes

$$H'(p_i, q_i) = H(p_i, q_i) + \sum_{a=1}^{m} \lambda_a \varphi_a(p_i, q_i) \qquad (13)$$

- let us assume that the consistency conditions

$$\dot{\varphi}_a(p_i, q_i) = \{H, \varphi_a\} + \sum_{b=1}^{m} \lambda_b \{\varphi_b, \varphi_a\} = 0 \qquad (14)$$

is fulfilled with each of the Poisson brackets above being zero:

$$\{\varphi_a, \varphi_b\} = \sum_{c=1}^{m} C_{abc} \varphi_c \qquad (15)$$

$$\{H, \varphi_a\} = \sum_{b=1}^{m} d_{ab} \varphi_b \qquad (16)$$

- since $\{\varphi_a, \varphi_b\} \equiv 0$, the Lagrange multipliers cannot be determined from (14).

• Consider a dynamical variable (observable) $f(p_i, q_i, t)$

$$\dot{f}(p_i, q_i, t) = \{H', f\} = \{H, f\} + \sum_{a=1}^{m} \lambda_a \{\varphi_a, f\} \qquad (17)$$

- the value of $f(p_i, q_i, t)$ after a short time interval δt:

$$f(p_i, q_i, t+\delta t) = f(p_i, q_i, t) + \delta t\, \dot{f}(p_i, q_i, t)$$

$$\overset{(17)}{=} f(p_i, q_i, t) + \delta t \left[\{H, f\} + \sum_{a=1}^{m} \lambda_a \{\varphi_a, f\}\right] \qquad (18)$$

- if now we consider different values for the arbitrary Lagrange multipliers, λ'_a,

$$f(p_i, q_i, t+\delta t) = f(p_i, q_i, t) + \delta t \left[\{H, f\} + \sum_{a=1}^{m} \lambda'_a \{\varphi_a, f\}\right], \qquad (19)$$

i.e.

$$\Delta_\lambda f(p_i, q_i, t+\delta t) = \delta t \sum_{a=1}^{m} (\lambda_a - \lambda'_a) \{\varphi_a, f\} \qquad (20)$$

- eq.(20) shows that the value of a dynamical variable at a time $t+\delta t$ does not depend only on the initial condition (its value at the time t), but also on the Lagrange multipliers.

- thus, a set of values of $f(p_i, q_i, t+\delta t)$, corresponding to all values of λ_a, describes THE SAME PHYSICAL STATE (the corresponding set of values is called an ORBIT), and all the values on the orbit are PHYSICALLY EQUIVALENT.

- due to the physical equivalence, one can make a choice of ONE value on the orbit (i.e ONE value of λ_a), by imposing subsidiary constraints

$$\chi_a(p_i, q_i) = 0 \qquad (21)$$

SUBSIDIARY constraints

- to determine λ_a uniquely, $\chi_a(p_i, q_i)$ have to satisfy the following requirements:

(i) consistency with e.o.m

$$\dot{\chi}_b = \{H, \chi_b\} + \sum_a \lambda_a \{\varphi_a, \chi_b\} = 0 \quad (22)$$

(ii) no. of constraints $\varphi_a = 0$ equal to no. of subsidiary constraints $\chi_a = 0$, i.e. m ($\Rightarrow \{\varphi_a, \chi_b\}$ square matrix $m \times m$).

(iii) $\det |\{\varphi_a, \chi_b\}| \neq 0$, (23)

i.e. integrability condition for (22).

With the above requirements fulfilled, λ_a are uniquely determined from (22).

- for <u>convenience</u>, one takes also

$$\{\chi_a, \chi_b\} = 0 \quad (24)$$

(though it is not necessary!)

<u>Summary</u>

- Phase space Γ $\quad 2n$ variables $(p_i, q_i, \; i = \overline{1,n})$.

- Constraints $\varphi_a = 0$ $\quad m$

s.t. $\{\varphi_a, \varphi_b\} = 0$

- Subsidiary constraints $\chi_a = 0$ $\quad m$

s.t. $\{\chi_a, \chi_b\} = 0$

and $\det |\{\varphi_a, \chi_b\}| \neq 0$.

$$\overset{(22)}{\Longrightarrow} \lambda_a = -\sum_{b=1}^{m} \{\varphi_a, \chi_b\}^{-1} \{H, \chi_b\} \text{ - uniquely determined! } (25)$$

- with χ_a fixed by (25), the equations of motion become:

$$\dot{f} = \{H, f\}_{DB}, \qquad (26)$$

where $\{\cdot, \cdot\}_{DB}$ - Dirac bracket,

$$\{f,g\}_{DB} = \{f,g\}\Big|_{\substack{\varphi_a=0 \\ \chi_b=0}} - \{f,\varphi_a\}\{\varphi_a,\chi_b\}^{-1}\{\chi_b,g\}\Big|_{\substack{\varphi_a=0 \\ \chi_b=0}}$$

- no. of INDEPENDENT variables

$$\underbrace{2n}_{\text{dim of phase space}} - \underbrace{2m}_{\text{constraints}} = 2(n-m), \qquad (26).$$

therefore the path integral has to be appropriately written over the $2(n-m)$ variables.

How to proceed?

- canonical transf. in the phase-space, s.t.

$$\chi_a(p_i, q_i) = p_a, \quad a = \overline{1,m} \qquad (27)$$

p_a - part of the CANONICAL MOMENTA

- find q_a, $a = \overline{1,m}$ - the COORDINATES canonically conjugated to p_a
- the rest of the phase-space variables (INDEPENDENT!)

$$p_i^*, \; q_i^* \qquad i = \overline{1, n-m} \text{ defining } \Gamma^* \qquad (28)$$

- then

$$(27) \Rightarrow \quad p_a = 0 \qquad (29)$$

$$(12) \Rightarrow \quad q_a = q_a(q^*, p^*) \qquad (30)$$

• The path integral for this system is

$$\int e^{i\int_0^t \left(\sum_{j=1}^{n-m} p_j^* \dot{q}_j^* - H^*(p_j^*, q_j^*)\right) d\tau} \prod_{\tau=0}^{t} \prod_{j=1}^{n-m} \frac{dq_j^* \, dp_j^*}{2\pi} \qquad (31)$$

$$\equiv \int e^{i\int_0^t \sum_{i=1}^{m} \left(p_i \dot{q}_i - H(p_i, q_i)\right) d\tau} \prod_{\tau=0}^{t} d\mu\left(p(\tau), q(\tau)\right), \qquad (32)$$

where the integration measure is

$$d\mu = \frac{1}{(2\pi)^{n-m}} \det\left|\{\chi_a, \varphi_b\}\right| \prod_{a=1}^{m} \delta(\chi_a)\,\delta(\varphi_a) \prod_{i=1}^{n} dq_i(\tau)\, dp_i(\tau) \qquad (33)$$

Remark: The expression (31) is written in terms of only the independent variables p_j^*, q_j^*, $j = \overline{1, n-m}$. This expression is in effect a <u>formal</u> one, since usually the constraints cannot be solved explicitly as in (29), (30). The equivalent expression (32) of the path integral is the one that is used in further manipulations.

- Show the equivalence of (32) to (31).

$$\{\chi_a, \varphi_b\} = \sum_{i=1}^{n} \left(\frac{\partial \chi_a}{\partial p_i} \frac{\partial \varphi_b}{\partial q_i} - \frac{\partial \chi_a}{\partial q_i} \frac{\partial \varphi_b}{\partial p_i} \right)$$

$$\overset{(27)}{=} \sum_{i=1}^{n} \left(\delta_{ai} \frac{\partial \varphi_b}{\partial q_i} - 0 \right) = \frac{\partial \varphi_b}{\partial q_a} \qquad (34)$$

$\Downarrow$ (33)

$$d\mu = \frac{1}{(2\pi)^{n-m}} \det \left| \frac{\partial \varphi_b}{\partial q_a} \right| \prod_{a=1}^{m} \delta(p_a)\, \delta(\varphi_a) \prod_{i=1}^{n} dq_i\, dp_i \qquad (35)$$

Note $\quad \delta(y(x))\, dy = \delta(x - x_0)\, dx \,, \quad y(x)\Big|_{x_0} = 0 \qquad (36).$

$$\Rightarrow \left| \frac{dy}{dx} \right| \delta(y(x))\, dx = \delta(x - x_0)\, dx \qquad (37)$$

$$\Rightarrow \prod_{a=1}^{m} \det \left| \frac{\partial \varphi_b}{\partial q_a} \right| \delta(\varphi_a)\, dq_a = \prod_{a=1}^{m} \delta\left(q_a - q_a(q^*, p^*)\right) dq_a \qquad (37)$$

$(y \to \varphi_a, \; x \to q_a)$

- thus, (35) becomes

$$d\mu = \prod_{a=1}^{m} \delta(p_a)\, \delta\left(q_a - q_a(q^*, p^*)\right) dq_a\, dp_a \prod_{j=1}^{n-m} \frac{dq_j^*\, dp_j^*}{2\pi} \qquad (38)$$

- introducing (38) into (32) and performing the δ-funct integrals over dq_a, dp_a, $a = \overline{1,m}$, one obtains the equivalent form (31)

q.e.d.

- Note: the expression (32) can be written also with integration over the Lagrange multipliers:

$$\int e^{i\int_0^t \left(\sum p_i \dot q_i - H(p_i q_i) - \sum_a \lambda_a \varphi_a\right)d\tau} \prod_{\tau=0}^{t} \det|\{\chi_a, \varphi_b\}| \frac{1}{(2\pi)^{n-m}}$$

$$\times \prod_{a=1}^{m} \delta(\chi_a) \prod_{i=1}^{n} dq_i\, dp_i \prod_b \frac{d\tau\, d\lambda_b}{2\pi}, \qquad (39)$$

since

$$\int e^{-i\sum_{i,a} \lambda_a(\tau_i)\varphi_a(p(\tau_i), q(\tau_i))\Delta\tau_i} \prod_{\tau=0}^{t} \frac{\Delta\tau_i\, d\lambda_b}{2\pi}$$

$$= \prod_{i,a} \delta\left(\varphi_a(p(\tau_i), q(\tau_i))\right) \xrightarrow[\Delta\tau_i \to 0]{} \prod_a \delta(\varphi_a) \qquad (40)$$

The expression (39) shows that the Lagrange multipliers, which have been introduced in order to take care of the constraints $\varphi_a = 0$, have to be integrated over in the path integral, since the observables should not depend on them.