Grassmann variables

- In QFT, Grassmann variables are the classical counterpart of anticommuting operators (used in the quantization of fermionic systems)

Grassmann algebra G_n

- An algebra with generators

$$\theta_1, \theta_2, \ldots \theta_n \tag{1}$$

satisfying the anticommutation relation:

$$\{\theta_i, \theta_k\} = \theta_i \theta_k + \theta_k \theta_i = 0 \tag{2}$$

- From (2), it follows immediately that

$$\theta_i^2 = 0 \quad , (\forall)\, i \tag{3}$$

$\Downarrow$

Grassmann variables are nilpotent

- Any element $f(\theta_1, .. \theta_n) \in G_n$ can be represented as

$$f(\theta_1 .. \theta_n) = f_0 + \sum_k f_1(k)\theta_k + \sum_{k_i} f_2(k_1, k_2)\theta_{k_1}\theta_{k_2} + \ldots + \sum_{k_i} f_n(k_1, \ldots k_n)\theta_{k_1}, \ldots \theta_{k_n} \tag{4}$$

- the form (4) is not unique
- uniqueness can be achieved if the coefficients $f_i(k_1, k_2, \ldots k_i)$ are taken to be antisymmetric with respect to the interchange of their arguments

- In particular, if a function $f(\theta)$ depends on only one Grassmann var., its Taylor expansion is:

$$f(\theta) = a + b\theta \qquad (5)$$

- **Derivatives**
  - since θ_i anticommute, the direction in which the derivatives act has to be specified:

LEFT DERIVATIVE:

$$\frac{\partial}{\partial\theta_i}(\theta_j\theta_k) = \frac{\partial\theta_j}{\partial\theta_i}\theta_k - \theta_j\frac{\partial\theta_k}{\partial\theta_i} = \delta_{ij}\theta_k - \delta_{ik}\theta_j \qquad (6)$$

RIGHT DERIVATIVE:

$$(\theta_j\theta_k)\frac{\partial}{\partial\theta_i} = \theta_j\frac{\partial\theta_k}{\partial\theta_i} - \frac{\partial\theta_j}{\partial\theta_i}\theta_k = \delta_{ik}\theta_j - \delta_{ij}\theta_k \qquad (7)$$

Remark: left and right derivatives of the same object are different, therefore the direction must be specified.

- derivatives with respect to Grassmann variables anticommute:

$$\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} = 0 \qquad (8)$$

$$\Downarrow$$

$$\left(\frac{\partial}{\partial\theta_i}\right)^2 = 0 \qquad (9)$$

- the conventional commutator between derivatives and coordinates becomes anticommutator:

$$\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \frac{\partial}{\partial\theta_i}\theta_j + \theta_j\frac{\partial}{\partial\theta_i} = \delta_{ij} \qquad (10)$$

- <u>Integration</u>
  - denote D – differentiation operation with respect to one Grassmann variable

    I – integration operation
  - the operators D and I have to satisfy:

$$ID = 0 , \qquad (11)$$

i.e. the integral of a total derivative must vanish if we ignore surface terms

$$DI = 0 , \qquad (12)$$

i.e. the differentiation of the integral must be zero, since the integral does not depend on the variable.

- due to (9),

$$D^2 = 0 , \qquad (13)$$

and putting together (11), (12) and (13) it follows that

INTEGRATION CAN NATURALLY BE IDENTIFIED WITH DIFFERENTIATION

$$I = D , \qquad (14)$$

i.e.

$$\int d\theta \, f(\theta) = \frac{\partial f(\theta)}{\partial \theta} \qquad (15)$$

$$\Downarrow$$

$$\left.\begin{aligned} \int d\theta &= 0 \\ \int d\theta \, \theta &= 1 \end{aligned}\right\} \qquad (16)$$

- One can see that from the properties

$$\{d\theta_i, d\theta_j\} = 0 \qquad (17)$$
$$\{\theta_i, d\theta_j\} = 0 \; ,$$

it follows automatically that $\int d\theta = 0$, since

$$\int d\theta_i \, d\theta_j = -\int d\theta_j \, d\theta_i \Rightarrow \int d\theta_i \, d\theta_j = 0 \; , \qquad (18)$$

which can come from defining

$$\int d\theta_i = 0 \; , \; \int d\theta_j = 0 \qquad (19)$$

- The integral $\int d\theta \, \theta = 1$ would be more natural to define as the square of the integration "volume". But in practical applications always the final infinite volume drops out, which justifies the normalization to 1 of the integral.

• Change of variables

$$\theta' = a\theta \quad , a \neq 0 \qquad (20)$$

- then

$$\int d\theta \, f(\theta) = \frac{\partial f(\theta)}{\partial \theta} = a \frac{\partial f\left(\frac{\theta'}{a}\right)}{\partial \theta'} = a \int d\theta' f\left(\frac{\theta'}{a}\right) \qquad (21)$$

$\Rightarrow$ exactly the opposite of what happens for usual variables (the Jacobian for Grassmann variables is the inverse of what one would expect for ordinary variables)

- for the case of many Grassmann variables:

$$\theta'_i = a_{ij}\theta_j \quad , \quad \det a_{ij} \neq 0 \qquad (22)$$

$$\Rightarrow \int d\theta_1 \ldots d\theta_n \, f(\theta_i) = (\det a_{ij}) \int d\theta'_1 \ldots d\theta'_n \, f(a^{-1}_{ij}\theta'_j) \qquad (23)$$

• Delta function

$$\delta(\theta) = \theta \qquad (24)$$

(i) $\int d\theta\, \delta(\theta) = \int d\theta\, \theta = 1 \qquad (25)$

(ii) for $f(\theta) = a + b\theta$,

$$\int d\theta\, \delta(\theta) f(\theta) = \int d\theta\, \theta\, (a + b\theta) = \int d\theta\, \theta\, a = \frac{\partial(\theta a)}{\partial\theta} = a = f(0) \qquad (26)$$

(iii) for $g(\theta) = a\theta$,

$$\delta(g(\theta)) = a\theta = a\,\delta(\theta) = \frac{\partial g(\theta)}{\partial\theta}\,\delta(\theta) \qquad (27)$$

(iv) integral representation for the delta function :

ζ - Grassmann variable

$$\int d\zeta\, e^{i\zeta\theta} = \int d\zeta\, (1 + i\zeta\theta) = \frac{\partial}{\partial\zeta}(1 + i\zeta\theta) = i\theta = i\,\delta(\theta) \qquad (28)$$

Grassmann algebra with involution

- complex conjugated elements are defined and included in the algebra

$$\theta,\ \theta^* , \qquad (29)$$

with the properties:

$$\begin{aligned} (\theta^*)^* &= \theta \\ (\theta_1\theta_2)^* &= \theta_2^*\,\theta_1^* \\ (\alpha\,\theta)^* &= \bar{\alpha}\,\theta^* \quad (\bar{\alpha} \text{ is complex conj. of } \alpha) \\ (\theta^*, \zeta^*) &= (\zeta, \theta) \quad \text{- inner product} \end{aligned} \qquad (30)$$

- anticommutators :

$$\{\theta_i, \theta_j\} = \{\theta_i^*, \theta_j\} = \{\theta_i^*, \theta_j^*\} = 0 \qquad (31)$$

$$\Downarrow$$

$$\theta_i^2 = 0 , \quad (\theta_i^*)^2 = 0 \quad , (\forall)\, i$$

- any function can be written as a <u>finite</u> polynomial
- obviously,

$$\{d\theta_i^*, d\theta_j\} = \{d\theta_i^*, d\theta_j^*\} = 0$$
$$\{\theta_i, d\theta_j^*\} = \{\theta_i^*, d\theta_j\} = \{\theta_i^*, d\theta_j^*\} = 0 , \qquad (32)$$

and

$$\int d\theta_i = \int d\theta_i^* = 0$$
$$\int d\theta_i\, \theta_i = \int d\theta_i^*\, \theta_i^* = 0 \qquad (33) .$$

- Evaluation of the Gaussian integral for Grassmann variables:

$$I = \int \prod_{i,j} d\theta_i^*\, d\theta_j\; e^{-(\theta_i^* M_{ij} \theta_j + c_i^* \theta_i + \theta_i^* c_i)} , \qquad (34)$$

where c_i, c_i^* are independent Grassman variables.
- by change of variables, one shows that

$$I = \det M_{ij}\; e^{c_i^* M_{ij}^{-1} c_j} , \qquad (35)$$

which is the same as the corresponding integral for ordinary variables, except the <u>POSITIVE</u> power of the determinant.

- Matrix representation of Grassmann algebra

  – for two Grassmann var, the algebra can be repr. by 4x4 matrices :

$$\theta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \theta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad (36)$$

  – for a Grassmann alg with n generators, the representation is in terms of $2^n \times 2^n$ matrices.

Some history :

- Hermann Grassmann (1809 - 1877)
  - exterior algebra (Grassmann alg. is the exterior algebra of a vector space)

- Use in supersymmetry :
  - Felix Berezin 1969 – graded algebra
  - Gol'fand and Likhtman , 1971
  - Volkov and Akulov , 1972
  - Wess and Zumino , 1973

Path integral quantization

Generating functional

- Path integral quantization of field theory relies on the concept of generating functional, from which all the Green functions of the theory are obtained
- In preparation for field theory quantization, we shall introduce the generating functionals in quantum mechanics.
- The prototype will be the well-studied (already) quantum harmonic oscillator, since quantum field theory can be viewed as an extension to an infinite number of degrees of freedom of quantum oscillator type.

- Consider a 1-dim harmonic oscillator which interacts with an external sourse:

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 + J(t) x \qquad (1)$$

$$S = \int_{t_i}^{t_f} dt \, L \qquad (2)$$

t_i - initial time
t_f - final time

- classical trajectory satisfies the eq. of motion

$$m \ddot{x}_c(t) + m\omega^2 x_c(t) - J(t) = 0 \qquad (3)$$

- transition amplitude (propagator) as path integral:

$$K(x_f, t_f; x_i, t_i) = \langle x_f, t_f | x_i, t_i \rangle = \int e^{\frac{i}{\hbar} S[x]} \prod_{\tau = t_i}^{t_f} \frac{dx(\tau)}{\sqrt{\frac{2\pi \hbar i}{m} d\tau}} \qquad (4)$$

- semiclassical approximation - gives exact result!

$$x(t) = x_c(t) + X(t) \qquad (5)$$

$$S[x(t)] = S[x_c + X] = S[x_c(t)] + \int dt\, X(t) \underbrace{\frac{\delta S[x]}{\delta x(t)}\Big|_{x=x_c}}_{\text{extremality condition}\;=\;0}$$

$$+ \frac{1}{2!}\int dt_1\, dt_2\, X(t_1)\, X(t_2) \frac{\delta^2 S[x]}{\delta x(t_1)\, \delta x(t_2)}\Big|_{x=x_c}$$

$$= S[x_c(t)] + \frac{1}{2}\int_{t_i}^{t_f} d\tau \left(m \dot{X}^2 - m\omega^2 X^2\right) \qquad (6)$$

$$\text{with } X(t_i) = X(t_f) = 0 \qquad (7)$$

- then

$$K(x_f, t_f; x_i, t_i) = e^{\frac{i}{\hbar} S[x_c(t)]} \int e^{\frac{i}{2\hbar}\int_{t_i}^{t_f} d\tau \left(m\dot{X}^2 - m\omega^2 X^2\right)} \prod_{\tau=t_i}^{t_f} \frac{dX(\tau)}{\sqrt{\frac{2\pi i\hbar}{m} d\tau}} \qquad (8)$$

- recall the result previously obtained:

$$K(x_f, t_f; x_i, t_i) = \left(\frac{m}{2\pi i \hbar (t_f - t_i)}\right)^{1/2} \left(\frac{\omega(t_f - t_i)}{\sin\omega(t_f - t_i)}\right)^{1/2} e^{\frac{i}{\hbar} S[x_c]} \qquad (9)$$

We shall focus on the path integral over the quantum fluctuations $X(\tau)$ in (8):

$$\int e^{\frac{i}{2\hbar} m \int_{t_i}^{t_f} d\tau \left(\dot{X}^2 - \omega^2 X^2\right)} \prod_{\tau=t_i}^{t_f} dX(\tau)$$

$$= \int e^{-\frac{im}{2\hbar}\int_{t_i}^{t_f} d\tau\, X(\tau)\left(\frac{d^2}{d\tau^2} + \omega^2\right) X(\tau)} \mathcal{D}X \qquad (10)$$

with the boundary conditions (7) and the notation

$$\mathcal{D}X = \prod_{\tau=t_i}^{t_f} dX(\tau) \qquad (11)$$

- we have seen (Lecture Notes, p. 17) that the Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-a x^2} = \sqrt{\frac{\pi}{a}}$$

generalizes for a $n \times n$ matrix

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \ldots dx_n \; e^{-\sum_{i,j} x_i A_{ij} x_j} = \left(\frac{\pi^n}{\det A} \right)^{1/2} \qquad (12)$$

- in the continuum limit (path integral), (12) generalizes further to

$$\int \mathcal{D}X \; e^{i \int_{t_i}^{t_f} d\tau\, X(\tau)\, \mathcal{O}(\tau)\, X(\tau)} = N \left[\det \mathcal{O}(t) \right]^{-1/2}, \qquad (13)$$

where $\mathcal{O}(t)$ is a Hermitian operator and N is a normalization constant.

- we show that (13) is indeed true for the harmonic oscillator:

- we analytically continue the integrals in (10) to complex time (to get rid of the oscillatory behaviour):

$$\tau \rightarrow \tau' = -is \,, \quad s \text{ real} \qquad (14)$$

$$\int \mathcal{D}X(\tau)\; e^{-\frac{im}{2\hbar} \int_{t_i}^{t_f} d\tau\, X(\tau) \left(\frac{d^2}{d\tau^2} + \omega^2 \right) X(\tau)}$$

$$\rightarrow \int \mathcal{D}X(s)\; e^{-\frac{m}{2\hbar} \int_{s_i}^{s_f} ds\, X(s) \left(-\frac{d^2}{ds^2} + \omega^2 \right) X(s)}$$

$$= N' \int \mathcal{D}X(s)\; e^{-\int_{s_i}^{s_f} ds\, X(s) \left(-\frac{d^2}{ds^2} + \omega^2 \right) X(s)}, \qquad (15)$$

where N' is a normalization factor appearing from the rescaling of the X variable at the last step of (15).

- boundary conditions $X(s_i) = X(s_f) = 0$ (16)

- comparing (15) to (13), it turns out that the "matrix" (operator) $\mathcal{O}(s)$ is

$$-\frac{d^2}{ds^2}+\omega^2 \qquad (17)$$

and its determinant is given by the product of its eigenvalues

$$\left(-\frac{d^2}{ds^2}+\omega^2\right)\psi_n(s) = \lambda_n \psi_n(s) , \qquad (18)$$

$$\det\left(-\frac{d^2}{ds^2}+\omega^2\right) = \prod_n \lambda_n \qquad (19)$$

- this typical problem of eigenvalues and eigenfunctions has the solution

$$\psi_n(s) = \sqrt{\frac{2}{(s_f-s_i)}}\ \sin\frac{n\pi(s-s_i)}{s_f-s_i} \qquad (20)$$

$$\lambda_n = \left(\frac{n\pi}{s_f-s_i}\right)^2+\omega^2 \qquad (21)$$

- thus,

$$\det\left(-\frac{d^2}{ds^2}+\omega^2\right) = \prod_{n=1}^{\infty}\lambda_n = \prod_{n=1}^{\infty}\left[\left(\frac{n\pi}{s_f-s_i}\right)^2+\omega^2\right] =$$

$$= \underbrace{\prod_{n=1}^{\infty}\left(\frac{n\pi}{s_f-s_i}\right)^2}_{} \ \underbrace{\prod_{n=1}^{\infty}\left[1+\left(\frac{\omega(s_f-s_i)}{n\pi}\right)^2\right]}_{}$$

$$= \underset{\text{constant}}{B}\ \frac{\sinh\omega(s_f-s_i)}{\omega(s_f-s_i)} \qquad (22)$$

- continuing (22) back to real time, we obtain

$$\det\left(-\frac{d^2}{ds^2}+\omega^2\right) \longrightarrow \det\left(\frac{d^2}{d\tau^2}+\omega^2\right) = A\ \frac{\sin\omega(t_f-t_i)}{\omega(t_f-t_i)} \qquad (23)$$

- comparing (9) with (15) + (23), we conclude that

$$\int \mathcal{D}X \, e^{\frac{im}{2\hbar}\int_{t_i}^{t_f} d\tau \left(\dot{X}^2 - \omega^2\right)} = N\left[\det\left(\frac{d^2}{d\tau^2}+\omega^2\right)\right]^{-1/2} \quad (24)$$

- explicitly writing the regularization of the oscillatory integrand:

$$\lim_{\varepsilon\to 0}\int \mathcal{D}X \, e^{-\frac{im}{2\hbar}\int_{t_i}^{t_f} d\tau \left(X(\tau)\left(\frac{d^2}{d\tau^2}+\omega^2 - i\varepsilon\right)X(\tau)\right)} = \lim_{\varepsilon\to 0} N\left[\det\left(\frac{d^2}{d\tau^2}+\omega^2-i\varepsilon\right)\right]^{-1/2} \quad (25)$$

• Time-ordered correlation functions

- recall the relation between Schrödinger picture states $|x\rangle$ and Heisenberg picture states $|x,t\rangle$:

$$|x,t\rangle = e^{\frac{i}{\hbar}Ht}|x\rangle, \quad (26)$$

(the state $|x,t\rangle$ does NOT DEPEND on time, but is the state of the system AT THE TIME t.)

s. t.

$$\langle x,t|x',t\rangle = \langle x|e^{-\frac{i}{\hbar}Ht}\, e^{\frac{i}{\hbar}Ht}|x'\rangle = \langle x|x'\rangle = \delta(x-x') \quad (27)$$

and

$$\int dx\, |x,t\rangle\langle x,t| = \int dx\, e^{\frac{i}{\hbar}Ht}|x\rangle\langle x|e^{-\frac{i}{\hbar}Ht} =$$

$$= e^{\frac{i}{\hbar}Ht}\left(\int dx\,|x\rangle\langle x|\right)e^{-\frac{i}{\hbar}Ht} = 1 \quad (28)$$

Note: The normalization of Heisenberg picture states and the completeness relation hold only at equal times!

- the coordinate operator in Schrödinger picture

$$\hat{x}|x\rangle = x|x\rangle$$

and in Heisenberg picture:

$$\hat{x}_H(t) = e^{\frac{i}{\hbar}Ht}\,\hat{x}\,e^{-\frac{i}{\hbar}Ht} \quad (29)$$

$$\hat{x}_H(t)|x,t\rangle = x|x,t\rangle \quad (30)$$

– starting from the basic path integral

$$\langle x_f, t_f | x_i, t_i \rangle = N \int \mathcal{D}x \, e^{\frac{i}{\hbar} S[x]} , \qquad (31)$$

let us evaluate the matrix element between the same initial and final states of the operator

$$\hat{x}_H(t_1)\, \hat{x}_H(t_2) \quad , \quad t_f \geq t_1 > t_2 \geq t_i \qquad (32)$$

i.e.

$$\langle x_f, t_f | \hat{x}_H(t_1)\, \hat{x}_H(t_2) | x_i, t_i \rangle =$$

$$= \int dx_1 dx_2 \; \langle x_f, t_f | \underbrace{\hat{x}_H(t_1) | x_1, t_1 \rangle}_{x_1 | x_1, t_1 \rangle} \langle x_1, t_1 | \underbrace{\hat{x}_H(t_2) | x_2, t_2 \rangle}_{x_2 | x_2, t_2 \rangle} \langle x_2 t_2 | x_i, t_i \rangle$$

from (30)

$$= \int dx_1 dx_2 \; x_1 x_2 \langle x_f, t_f | x_1, t_1 \rangle \langle x_1, t_1 | x_2, t_2 \rangle \langle x_2, t_2 | x_i, t_i \rangle$$

$$= \int dx_1\, dx_2\, x_1 x_2 \int e^{\frac{i}{\hbar}\int_{t_1}^{t_f} L\, d\tau} \prod_{\tau = t_1}^{t_f} \frac{dx(\tau)}{\sqrt{\frac{2\pi\hbar i}{m} d\tau}} \int e^{\frac{i}{\hbar}\int_{t_2}^{t_1} L\, d\tau} \prod_{\tau = t_2}^{t_1} \frac{dx(\tau)}{\sqrt{\frac{2\pi\hbar i}{m} d\tau}} \times$$

$$\times \int e^{\frac{i}{\hbar}\int_{t_i}^{t_2} L\, d\tau} \prod_{\tau = t_i}^{t_2} \frac{dx(\tau)}{\sqrt{\frac{2\pi\hbar i}{m} d\tau}}$$

$$= \int x(t_1)\, x(t_2)\, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} L\, d\tau} \prod_{\tau = t_i}^{t_f} \frac{dx(\tau)}{\sqrt{\frac{2\pi\hbar i}{m} d\tau}} = N \int \mathcal{D}x \;\; x(t_1) x(t_2)\, e^{\frac{i}{\hbar} S[x]} ,$$

$$\langle x_f, t_f | \hat{x}_H(t_1)\, \hat{x}_H(t_2) | x_i, t_i \rangle_{t_1 > t_2} = N \int \mathcal{D}x \;\; x(t_1) x(t_2)\, e^{\frac{i}{\hbar} S[x]} \qquad (33)$$

where we identified

$$x_1 = x(t_1) \quad , \quad x_2 = x(t_2) \qquad (34)$$

– similarly we obtain, for $t_2 > t_1$,

$$\langle x_f, t_f | \hat{x}_H(t_2)\, \hat{x}_H(t_1) | x_i, t_i \rangle = N \int \mathcal{D}x \;\; x(t_2) x(t_1)\, e^{\frac{i}{\hbar} S[x]} \qquad (35)$$

$$= N \int \mathcal{D}x \;\; x(t_1)\, x(t_2)\, e^{\frac{i}{\hbar} S[x]}$$

since $x(t_1)$ and $x(t_2)$ commute (they are not operators!)

- thus, the path integral incorporates naturally the TIME ORDERING, since the evolution takes place succesivelly from a past state to a future state (see also the correlation fct with Wiener measure, Lecture Notes, p. 13-14):

$$\boxed{\langle x_f, t_f | T\left(\hat{x}_H(t_1)\hat{x}_H(t_2)\right) | x_i, t_i\rangle = N \int \mathcal{D}x \; x(t_1)x(t_2) e^{\frac{i}{\hbar}S[x]},} \quad (36)$$

where

$$\underbrace{T\left(\hat{x}_H(t_1)\hat{x}_H(t_2)\right)}_{\text{time-ordered product of operators}} = \theta(t_1-t_2)\hat{x}_H(t_1)\hat{x}_H(t_2) + \theta(t_2-t_1)\hat{x}_H(t_2)\hat{x}_H(t_1) \quad (37)$$

- in general, for any set of operators,

$$\langle x_f, t_f | T\left(\mathcal{O}_1(\hat{x}_H(t_1)) \dots \mathcal{O}_n(\hat{x}_H(t_n))\right) | x_i, t_i\rangle = N \int \mathcal{D}x \; \mathcal{O}_1(x(t_1)) \dots \mathcal{O}_n(x(t_n)) \, e^{\frac{i}{\hbar}S[x]} \quad (38).$$

- <u>Correlation functions in definite states</u>
  - transitions between physical states
  - find the transition amplitude for a system in the state $|\psi_i\rangle$ at time t_i to make a transition to the state $|\psi_f\rangle$ at time t_f

$$\langle \psi_f | \psi_i \rangle = \int dx_f \, dx_i \, \langle \psi_f | x_f, t_f\rangle \underbrace{\langle x_f, t_f | x_i, t_i\rangle} \langle x_i, t_i | \psi_i\rangle$$

$$= N \int dx_f \, dx_i \, \psi_f^*(x_f, t_f)\, \psi_i(x_i, t_i) \int \mathcal{D}x \, e^{\frac{i}{\hbar}S[x]}, \quad (39)$$

where $\psi(x,t) = \langle x, t | \psi\rangle$