

Plasma in thermal equilibrium

Start the discussion from **equilibrium statistical mechanics**

Let there be N particles in plasma $N/2$ electrons, $N/2$ ions.
Assume that the plasma is in thermal equilibrium at temperature T .

The probability to find the particles in locations $(\mathbf{r}_1, \dots, \mathbf{r}_N)$ is given by the **Gibbs distribution**

$$D(\mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{1}{Z} \exp\left(-\frac{\sum_k \sum_{i>k} W_{ik}}{k_B T}\right)$$

$$Z = \int \exp\left(-\frac{\sum_k \sum_{i>k} W_{ik}}{k_B T}\right) d^3 r_1 \dots d^3 r_N$$

partition function

$$W_{ik} = \frac{q_i q_k}{4\pi\epsilon_0 |\mathbf{r}_i - \mathbf{r}_k|} + \varphi_{ext}$$

Coulomb potential of plasma particles potential of external fields

The probability to find particle 1 at \mathbf{r}_1 is

$$F_1(\mathbf{r}_1) = \int D d^3 r_2 \dots d^3 r_N$$

If there are no external forces, $F_1 = 1/V$ (V is the volume)

The probability to find particle 1 at \mathbf{r}_1 and particle 2 at \mathbf{r}_2 is

$$F_2(\mathbf{r}_1, \mathbf{r}_2) = \int D d^3 r_3 \dots d^3 r_N$$

etc.

$$F_s(\mathbf{r}_1, \dots, \mathbf{r}_s) = \int D d^3 r_{s+1} \dots d^3 r_N$$

F_1, \dots, F_s are called **reduced distribution functions**

At the limit of non-interacting particles: $W_{ik} \rightarrow 0$

$$F_s \rightarrow F_1(\mathbf{r}_1) F_1(\mathbf{r}_2) \dots F_1(\mathbf{r}_s) = 1/V^s$$

Mayer cluster expansion for interacting particles (notation $r_1 \rightarrow 1$)

$$F_2(1, 2) = [1 + P_{12}(1, 2)]F_1(1)F_1(2)$$

$$F_3(1, 2, 3) = [1 + P_{12}(1, 2) + P_{12}(2, 3) + P_{12}(1, 3) + T_{123}(1, 2, 3)] \times F_1(1)F_1(2)F_1(3)$$

↙ ↘
↙ ↘
 pair correlation functions three-particle correlation function

At plasma limit: $\Lambda \gg 1$ the Coulomb interactions are weak \Rightarrow

$$T_{123} \ll P_{12} \ll 1$$

and we need to consider pair correlations only (a big relief compared to ordinary gases where higher correlations are important)

P is symmetric $P_{12}(1, 2) = P_{12}(|\mathbf{r}_1 - \mathbf{r}_2|)$

Note that the complete Gibbs distribution depends also on velocity

$$D^*(\mathbf{r}_1, \dots, \mathbf{r}_N, \mathbf{v}_1, \dots, \mathbf{v}_N) = \frac{1}{Z^*} \exp\left(-\frac{\sum_k \sum_{i>k} W_{ik}}{k_B T}\right) \exp\left(-\frac{\sum_i \frac{1}{2} m_i v_i^2}{k_B T}\right)$$

The velocity correlations become important in relativistic plasmas

We neglect them here.

Differentiate F_s with respect to \mathbf{r}_1 , set $s = 2$ and assume $T_{123} \ll P_{12} \Rightarrow$

$$\frac{\partial P_{12}}{\partial \mathbf{r}_1} + \frac{1}{4\pi\epsilon_0 k_B T} \frac{\partial}{\partial \mathbf{r}_1} \left(\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \right) + \frac{1}{4\pi\epsilon_0 k_B T} \sum_{\alpha} \frac{N_{\alpha}}{V} \int [P_{12}(2, \alpha) + P_{12}(1, \alpha)] \frac{\partial}{\partial \mathbf{r}_1} \left(\frac{q_1 q_{\alpha}}{|\mathbf{r}_1 - \mathbf{r}_{\alpha}|} \right) d^3 r_{\alpha} = 0.$$

Number of particles of species α

and here is the Debye shielding again

This can be solved by Fourier transformation yielding

$$P_{12}(|\mathbf{r}_1 - \mathbf{r}_2|) = -\frac{q_1 q_2}{4\pi\epsilon_0 k_B T} \frac{\exp(-|\mathbf{r}_1 - \mathbf{r}_2|/\lambda_D)}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

$P_{12} \ll 1$ is valid if $|\mathbf{r}_1 - \mathbf{r}_2| > \lambda_D$

The Mayer expansion is valid also inside the Debye sphere, where

$$P_{12} \propto 1/|\mathbf{r}_1 - \mathbf{r}_2|$$

as long as the distance $|\mathbf{r}_1 - \mathbf{r}_2|$ is larger than the average particle distance at temperature T

Plasma is very good ideal gas

$$p = nk_B T + O\left(\frac{1}{\Lambda}\right)$$

↙ ↘
 non-interacting particles weak pair correlations

Klimontovich equation approach

In reality plasma is seldom (if ever) in thermal equilibrium. There are different ways to find useful plasma equations independent of thermal equilibrium

Start by finding the exact density of particles in the 6D phase space (\mathbf{r}, \mathbf{v})

Consider a single particle whose orbit in the phase space is $(\mathbf{R}_1(t), \mathbf{V}_1(t))$

The "density" of this particle is Dirac's delta functions

$$N(\mathbf{r}, \mathbf{v}, t) = \delta[\mathbf{r} - \mathbf{R}_1(t)]\delta[\mathbf{v} - \mathbf{V}_1(t)]$$

i.e., at a given time the particle is at one point both in \mathbf{r} and \mathbf{v}

Sum over all particles of species α gives the density function N_α

That consists of a VERY large sum of δ functions!

Write the equation of motion under the Lorentz force for **all particles** and sum over all particles of species α

→ the **Klimontovich equation** for N_α

$$\frac{\partial N_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial N_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial N_\alpha}{\partial \mathbf{v}} = 0$$

This is still a very detailed description and the integration of the Klimontovich equation involves integration of a large number of δ functions.

We can take **ensemble averages** of N_α and the Klimontovich equation

Denote the ensemble average of $N_\alpha(\mathbf{r}, \mathbf{v}, t)$ by $f_\alpha(\mathbf{r}, \mathbf{v}, t)$

What does this mean?

Prepare an infinite number of realizations of plasma according to some prescription, e.g., all in equal temperature and each with a test charge

at a given point, take average over all these realizations

→ f_α consistent with Debye shielding

This procedure leads to the **Vlasov equation**

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0$$

Note that the steps taken above are not trivial, in particular the calculation of the derivatives of the δ functions!

Liouville equation approach

Instead of starting from individual particles and δ functions we can also start from a general distribution function in a $6N$ dimensional phase space

$$F(\mathbf{r}_1, \dots, \mathbf{r}_N; \mathbf{v}_1, \dots, \mathbf{v}_N; t)$$

with normalization $\int F d^3r_1 \dots d^3r_N d^3v_1 \dots d^3v_N = 1$

For a plasma of $N/2$ electrons and $N/2$ ions in thermodynamic equilibrium this is the Gibbs distribution, i.e., $F = D$.

Again F contains all information of all particles and is thus quite impractical. As in the thermodynamical equilibrium case define again reduced distribution functions as

$$f_{\alpha}^{(1)}(\mathbf{r}_1, \mathbf{v}_1, t) = V \int F d^3r_2 \dots d^3r_N d^3v_2 \dots d^3v_N$$

$$f_{\alpha\beta}^{(2)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}_1, \mathbf{v}_2, t) = V^2 \int F d^3r_3 \dots d^3r_N d^3v_3 \dots d^3v_N \quad \text{etc.}$$

V is the volume where F is nonzero for all $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$

Statistical physics tells us that F satisfies the **Liouville equation**

$$\frac{\partial F}{\partial t} + \sum_{i=1}^N \left(\frac{\partial F}{\partial \mathbf{r}_i} \cdot \mathbf{v}_i + \frac{\partial F}{\partial \mathbf{v}_i} \cdot \mathbf{a}_i^T \right) = 0 \quad \text{recall: } \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\mathbf{r}}{dt} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{d\mathbf{v}}{dt} \cdot \frac{\partial}{\partial \mathbf{v}}$$

acceleration by all forces and interactions

Integrate this over all coordinates except $(\mathbf{r}_1, \mathbf{v}_1) \Rightarrow$

$$\frac{\partial f_{\alpha}^{(1)}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_{\alpha}^{(1)}}{\partial \mathbf{r}_1} + V \int \mathbf{a}_i^T \cdot \frac{\partial F}{\partial \mathbf{v}_1} d^3r_2 \dots d^3r_N d^3v_2 \dots d^3v_N = 0$$

Here the total number of particles is assumed to be conserved
If there are external forces only, we get again the Liouville equation

$$\frac{\partial f_{\alpha}^{(1)}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_{\alpha}^{(1)}}{\partial \mathbf{r}_1} + \mathbf{a}_1^E \cdot \frac{\partial f_{\alpha}^{(1)}}{\partial \mathbf{v}_1} = 0$$

The total interaction term is

$$\mathbf{a}_1^E \cdot \frac{\partial f_{\alpha}^{(1)}}{\partial \mathbf{v}_1} + \sum_{\beta} \int \mathbf{a}_{1\beta} \cdot \frac{\partial}{\partial \mathbf{v}_1} f_{\alpha\beta}^{(2)}(\mathbf{r}_1, \mathbf{r}_{\beta}, \mathbf{v}_1, \mathbf{v}_{\beta}, t) d^3r_{\beta} d^3v_{\beta}$$

Interparticle interactions

We have reached a chain of equations where the equation of motion for the single-particle distribution function depends on the two-particle distribution. Writing a similar equation for two-particle distribution, we get dependence of three-particle distribution and so on. This is known as the **BBGKY hierarchy**

This chain becomes soon intractable and some physical insight is required to truncate it. We do it at the level of $f^{(2)}$. If the interparticle interactions were **strong** and of **short range** we would end up with **the Boltzmann equation**

$$\frac{df_{\alpha}^{(1)}}{dt} \equiv \frac{\partial f_{\alpha}^{(1)}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_{\alpha}^{(1)}}{\partial \mathbf{r}_1} + \mathbf{a}_1^E \cdot \frac{\partial f_{\alpha}^{(1)}}{\partial \mathbf{v}_1} = \left(\frac{\partial f_{\alpha}^{(1)}}{\partial t} \right)_c$$



But in plasma the interactions are **long-range** (Coulomb) and **weak**! Fortunately the combined effect of remote charges is, on the average, stronger than the acceleration by the nearest neighbour.

For an individual particle the average acceleration $\langle \mathbf{a}^{int} \rangle$ is similar to the acceleration by external (Coulomb) forces \mathbf{a}^E

Thus we can write for the acceleration $\mathbf{a}_1 = \mathbf{a}_1^E + \langle \mathbf{a}^{int} \rangle$ and take into account the binary collisions, if needed, in the form

$$\left(\frac{\partial f_{\alpha}^{(1)}}{\partial t} \right)_c = - \sum_{\beta} \int (\mathbf{a}_{1\beta} - \langle \mathbf{a}_{1\beta}^{int} \rangle) \cdot \frac{\partial}{\partial \mathbf{v}_1} f_{\alpha\beta}^{(2)} d^3 r_{\beta} d^3 v_{\beta}$$

this collision integral is in general rather complicated and requires use of approximative methods

Assuming that the only external force is the Lorentz force we finally have the **Boltzmann equation of plasma physics**

$$\frac{\partial f_{\alpha}^{(1)}}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_{\alpha}^{(1)}}{\partial \mathbf{r}_1} + \frac{q_{\alpha}}{m_{\alpha}} \langle \mathbf{E} + \mathbf{v}_1 \times \mathbf{B} \rangle \cdot \frac{\partial f_{\alpha}^{(1)}}{\partial \mathbf{v}_1} = \left(\frac{\partial f_{\alpha}^{(1)}}{\partial t} \right)_c$$

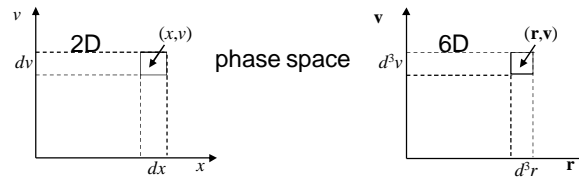
Of course, also Maxwell's equations must be valid at the same average (macroscopic) level

$$\nabla \cdot \langle \mathbf{E} \rangle = \frac{\rho}{\epsilon_0}$$

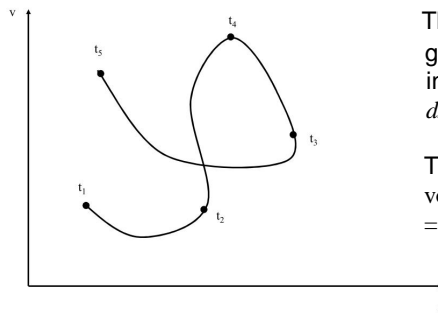
$$\nabla \times \langle \mathbf{B} \rangle = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \langle \mathbf{E} \rangle}{\partial t}$$

Note that the integral of F was normalized to 1. We get the familiar normalization by defining $f_{\alpha} = (N_{\alpha}/V) f_{\alpha}^{(1)}$

Distribution function



A plasma particle (i) is at time t in location $\mathbf{r}_i(t)$ and has velocity $\mathbf{v}_i(t)$



The **distribution function** $f(\mathbf{r}, \mathbf{v}, t)$ gives the particle number density in the (\mathbf{r}, \mathbf{v}) phase space element $dx dy dz dv_x dv_y dv_z$ at time t

The units of f :
 $\text{volume}^{-1} \times (\text{volume of velocity space})^{-1}$
 $= \text{s}^3 \text{m}^{-6}$

Normalization: $\int_V \int_{\mathbf{v}} f(\mathbf{r}, \mathbf{v}, t) d^3r d^3v = N$ ← total number of particles

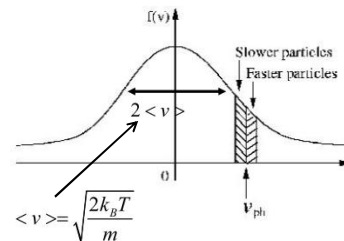
Sometimes (in particular in mathematical physics) the distribution function is normalized to 1.

Average density: $\langle n \rangle = N/V$; density at location \mathbf{r} : $n(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{v}, t) d^3v$

Example: **Maxwellian distribution**

$$f(v) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right)$$

$n = \langle n \rangle$

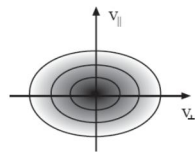
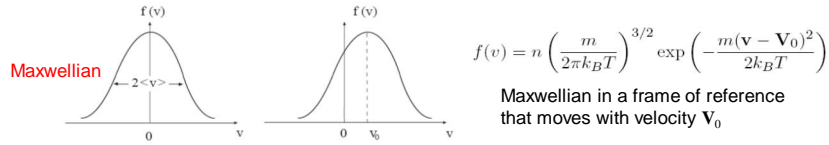


Exercise: Integrate this over the 3D velocity space to get n

Hint: $\int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi}$

$\langle v \rangle = \sqrt{\frac{2k_B T}{m}}$
 thermal velocity
 (velocity spread)

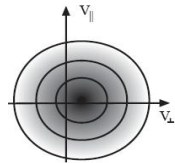
Examples of distribution functions



Anisotropic (pancake) distribution ($v_{\parallel} \parallel \mathbf{B}$)

$$f(v_{\perp}, v_{\parallel}) = \frac{n}{T_{\perp} T_{\parallel}^{1/2}} \left(\frac{m}{2\pi k_B} \right)^{3/2} \exp \left(-\frac{mv_{\perp}^2}{2k_B T_{\perp}} - \frac{mv_{\parallel}^2}{2k_B T_{\parallel}} \right)$$

Can also be **cigar-shaped** (elongated in the direction of \mathbf{B})

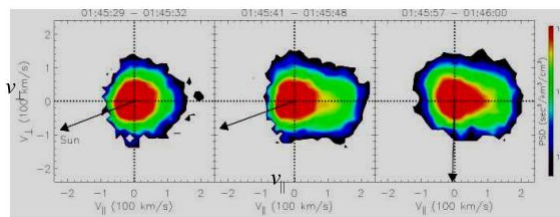


Drifting Maxwellian

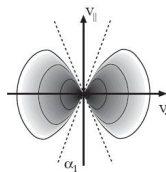
$$f(v_{\perp}, v_{\parallel}) = \frac{n}{T_{\perp} T_{\parallel}^{1/2}} \left(\frac{m}{2\pi k_B} \right)^{3/2} \exp \left(-\frac{m(\mathbf{v}_{\perp} - \mathbf{v}_{0\perp})^2}{2k_B T_{\perp}} - \frac{mv_{\parallel}^2}{2k_B T_{\parallel}} \right)$$

Magnetic field-aligned beam (e.g., particles causing the aurora):

$$f(v_{\perp}, v_{\parallel}) = \frac{n}{T_{\perp} T_{\parallel}^{1/2}} \left(\frac{m}{2\pi k_B} \right)^{3/2} \exp \left(-\frac{mv_{\perp}^2}{2k_B T_{\perp}} - \frac{m(v_{\parallel} - v_{0\parallel})^2}{2k_B T_{\parallel}} \right)$$



Loss-cone distribution in a magnetic bottle:



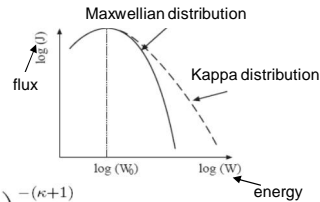
In reality plasma distributions are not really Maxwellian.

The next approximation: Maxwellian with a high-energy tail

The kappa distribution

The tail follows a **power law**

$$f(W) \propto (W_0/W)^\kappa$$



$$f_\kappa(W) = n \left(\frac{m}{2\pi\kappa W_0} \right)^{3/2} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - 1/2)} \left(1 + \frac{W}{\kappa W_0} \right)^{-(\kappa+1)}$$

$$[f_\kappa] = m^{-6} s^3$$

↑ ↑
Γ-function energy at the peak of the distribution

Observed particle distributions often resemble kappa distributions; a signature that non-thermal acceleration has taken place somewhere

Distribution as a function of energy

Consider a Maxwellian distribution

$$f(v) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{W}{k_B T}\right) \quad \text{where} \quad W = mv^2/2$$

How to replace $f(v)$ with a function of W ?

$$g(W) dW = f(v) d^3v \quad ; \quad d^3v = 4\pi v^2 dv \quad ; \quad dW = mv dv$$

$$\Rightarrow g(W) = 4\pi \left[\frac{2(W - U)}{m^3} \right]^{1/2} f(v) \quad \text{(U is here just a placeholder for a velocity independent potential)}$$

However, distribution functions are not measured directly. The measured quantity is the particle flux to the detector

$J(W, \alpha, \mathbf{r})$ is the **differential particle flux** per unit area for given energy, pitch-angle and location.

The particle flux through the surface is $n v_n$
↑
 velocity normal to the surface

Consider particles in the velocity interval dv and arrive in the solid angle $d\Omega$
 Then the number density of particles moving with velocity v in the unit
 phase space volume is $dn = f v^2 dv d\Omega$

Multiplying this with v , we get the for the differential flux the relation

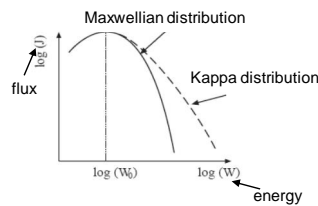
$$J(W, \alpha, \mathbf{r}) dW d\Omega = f(v_{\parallel}, v_{\perp}, \alpha, \mathbf{r}) v^3 dv d\Omega$$

As $dW = mv dv$

we have the relationship between the differential flux and distribution function

$$J(W, \alpha, \mathbf{r}) = \frac{v^2}{m} f(v_{\parallel}, v_{\perp}, \alpha, \mathbf{r})$$

Recall the representation
 of Maxwellian and the
 kappa distribution as flux
 vs. energy



Vlasov and Boltzmann equations equation(s) of motion for f

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 \quad \text{Vlasov equation (VE)}$$

Compare with the **Boltzmann equation** in statistical physics (BE)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t} \right)_c$$

Boltzmann derived $(\partial f / \partial t)_c$ for strong short-range collisions

In plasmas most collisions are long-range small-angle collisions.
 They are taken care by the average Lorentz force term



Ludwig Boltzmann

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t} \right)_c$$

large-angle collisions only
e.g., charge vs. neutral

VE is often called **collisionless Boltzmann equation**

(M. Rosenbluth: actually a Boltzmann-less collision equation!)

Velocity moments of f

$$\int f d^3v; \quad \int \mathbf{v}f d^3v; \quad \int \mathbf{v}\mathbf{v}f d^3v$$

$$n(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{v}, t) d^3v \quad \text{Density is the zeroth moment; } [n] = \text{m}^{-3}$$

The first moment:

$$\Gamma_\alpha(\mathbf{r}, t) = \int \mathbf{v}f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3v \quad \text{Particle flux; } [\Gamma] = \text{m}^{-2} \text{s}^{-1}$$

$$\mathbf{V}_\alpha(\mathbf{r}, t) = \frac{\int \mathbf{v}f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3v}{\int f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3v} \quad \text{Average velocity = flux/density, } [V] = \text{m s}^{-1}$$

DO NOT EVER MIX $\mathbf{V}(\mathbf{r}, t)$ and $\mathbf{v}(t)$!!

$$\mathbf{J}_\alpha(\mathbf{r}, t) = q_\alpha \Gamma_\alpha = q_\alpha n_\alpha \mathbf{V}_\alpha \quad \text{Electric current density, } [J] = \text{C m}^{-2} \text{s}^{-1} = \text{A m}^{-2}$$

Pressure and temperature

from the second velocity moments

$$\text{Pressure tensor} \quad \mathcal{P}_\alpha(\mathbf{r}, t) = m_\alpha \int \underbrace{(\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)}_{\text{dyadic product} \rightarrow \text{tensor}} f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3v$$

If $\mathcal{P}_\alpha = p_\alpha \mathcal{I}$ where \mathcal{I} is the unit tensor, we find the scalar pressure

$$p_\alpha = \frac{m_\alpha}{3} \int (\mathbf{v} - \mathbf{V}_\alpha)^2 f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3v = n_\alpha k_B T_\alpha \quad \text{introducing the temperature}$$

$$\text{Assume } \mathbf{V} = 0: \quad \frac{3}{2} k_B T_\alpha(\mathbf{r}, t) = \frac{m_\alpha}{2} \frac{\int v^2 f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3v}{\int f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3v} \quad T \propto \langle \text{K.E.} \rangle$$

Thus we can calculate a "temperature" also in non-Maxwellian plasma!

Magnetic pressure
(i.e. magnetic energy density) $B^2/2\mu_0$

$$\text{Plasma beta} \quad \beta = \frac{2\mu_0 \sum_\alpha n_\alpha k_B T_\alpha}{B^2} \quad \begin{array}{l} \beta \ll 1 \quad \mathbf{B} \text{ dominates over plasma} \\ \beta \gg 1 \quad \text{plasma dominates over } \mathbf{B} \end{array}$$

thermal pressure / magnetic pressure

3rd velocity moment \rightarrow heat flux (temperature x velocity), etc. to higher orders...

Macroscopic plasma description

Macroscopic plasma theories are **fluid** theories at different levels

- single fluid (magnetohydrodynamics MHD)
- two-fluid (multifluid, separate equations for electron and ion fluids)
- hybrid: fluid electrons with (macro)particle ions

Macroscopic equations can be obtained by taking velocity moments of Boltzmann / Vlasov equations

$$\int \mathbf{v}^n \left(\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} \right) d^3v = \int \left(\frac{\partial f_\alpha}{\partial t} \right)_c d^3v$$

↑ ↑
order n order $n+1$

Taking the n^{th} moment of BE/VE introduces terms of order $n+1$!
 This leads to an open chain of equations that must be terminated by applying some form of physical intuition (again!).

The collision integrals can be very tricky!

Continuity equation

We start from the Boltzmann equation

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left(\frac{\partial f_\alpha}{\partial t} \right)_c$$

and calculate its zeroth velocity moment.

- The first term gives trivially the time derivative of density.
- For physically relevant distributions ($f_\alpha \rightarrow 0$ when $v \rightarrow \infty$) the force term integrates to zero (exercise)
- In absence of ionizing or recombining collisions, the collision integral is zero

$$\int \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} d^3v = \nabla \cdot \int \mathbf{v} f_\alpha d^3v = \nabla \cdot (n_\alpha \mathbf{V}_\alpha) \quad \begin{array}{l} \text{exercise} \\ \text{here we get a first moment term} \end{array}$$

$$\Rightarrow \boxed{\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{V}_\alpha) = 0} \quad \text{Continuity equation}$$

Multiplying by mass or charge we get the continuity eqs for these

$$\frac{\partial \rho_{m\alpha}}{\partial t} + \nabla \cdot (\rho_{m\alpha} \mathbf{V}_\alpha) = 0 \quad \frac{\partial \rho_{q\alpha}}{\partial t} + \nabla \cdot \mathbf{J}_\alpha = 0$$

General form of **conservation law for F** :
 (\mathbf{G} is the flux of F)

$$\boxed{\frac{\partial F}{\partial t} + \nabla \cdot \mathbf{G} = 0}$$

Momentum equation (equation of motion)

To calculate the first moment, multiply BE by momentum $m_\alpha \mathbf{v}$ and integrate

$$\begin{aligned} \Rightarrow \text{exercise} \quad n_\alpha m_\alpha \frac{\partial \mathbf{V}_\alpha}{\partial t} + n_\alpha m_\alpha \mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha - n_\alpha q_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}) + \nabla \cdot \mathcal{P}_\alpha \\ = m_\alpha \int \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c d^3v. \end{aligned}$$

This tells how momentum is transported in plasma

Now the convective derivative of \mathbf{V} $\mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha$ and the pressure tensor \mathcal{P}_α are second moments

The electric and magnetic fields must fulfill Maxwell's equations $\nabla \cdot \mathbf{E} = \sum_\alpha \frac{n_\alpha q_\alpha}{\epsilon_0} + \frac{\rho_{ext}}{\epsilon_0}$

($\rho_{ext}, \mathbf{J}_{ext}$) are external sources $\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \sum_\alpha n_\alpha q_\alpha \mathbf{V}_\alpha + \mu_0 \mathbf{J}_{ext}$

Note that the collision integral can be non-zero, because collisions transfer momentum between different particle species!

One approximation of the collision integral is $m_\alpha \int \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_c d^3v \approx - \sum_\beta m_\alpha n_\alpha (\mathbf{V}_\alpha - \mathbf{V}_\beta) \langle \nu_{\alpha\beta} \rangle$

Energy equation

Calculate the second moment (multiply by $\mathbf{v}\mathbf{v}$, and integrate; rather tedious) \rightarrow **heat transfer equation** (conservation of energy)

The result can be written as (after a number of simplifying approximations):

$$\begin{aligned} \text{Total time derivative } d/dt = \partial/\partial t + \mathbf{V} \cdot \nabla \text{ of temperature} \quad \text{isotropic part of pressure tensor } p_\alpha \mathcal{I} \\ \frac{3}{2} n_\alpha k_B \left(\frac{\partial T_\alpha}{\partial t} + \mathbf{V}_\alpha \cdot \nabla T_\alpha \right) + p_\alpha \nabla \cdot \mathbf{V}_\alpha \\ = -\nabla \cdot \mathbf{H}_\alpha - (\mathcal{P}'_\alpha \cdot \nabla) \cdot \mathbf{V}_\alpha + \frac{\partial}{\partial t} \left(\frac{n_\alpha m_\alpha V_\alpha^2}{2} \right)_c \quad \text{kinetic energy changes due to collisions} \\ \text{effects of pressure anisotropies} \end{aligned}$$

The scalar pressure and temperature have been related in the calculation through $p_\alpha = n_\alpha k_B T_\alpha$ (plasma is almost ideal gas; explain why!)

\mathbf{H} is the **heat flux** (3rd moment) \rightarrow go to the next order, etc., etc.

To close the chain some equation relating the variables must be introduced. If heat flux can be neglected, the closure can be done at this level by, e.g., introducing an equation of state.

Equations of MHD

To derive MHD equations we first define single fluid variables as

$$\begin{aligned} \rho_m(\mathbf{r}, t) &= \sum_{\alpha} n_{\alpha} m_{\alpha} && \text{mass density} \\ \rho_q(\mathbf{r}, t) &= \sum_{\alpha} n_{\alpha} q_{\alpha} = e(n_i - n_e) && \text{charge density} \\ \mathbf{V}(\mathbf{r}, t) &= \frac{\sum_{\alpha} n_{\alpha} m_{\alpha} \mathbf{V}_{\alpha}}{\sum_{\alpha} n_{\alpha} m_{\alpha}} && \text{macroscopic (fluid) velocity} \\ \mathbf{J}(\mathbf{r}, t) &= \sum_{\alpha} n_{\alpha} q_{\alpha} \mathbf{V}_{\alpha} && \text{current density} \\ \mathcal{P}(\mathbf{r}, t) &= \sum_{\alpha} \mathcal{P}_{\alpha}^{CM}(\mathbf{r}, t) && \text{total pressure tensor} \end{aligned}$$

where $\mathcal{P}_{\alpha}^{CM}(\mathbf{r}, t) = m_{\alpha} \int (\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V}) f_{\alpha} d^3v$
are the pressure tensors of each species in centre of mass (CM) frame

By summing up the continuity eqs. for different species we get the **continuity eqs.** for mass and charge

$$\begin{aligned} \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) &= 0 \\ \frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{J} &= 0 \end{aligned}$$

The **momentum transfer equation** is the sum of the corresponding equations of each species (exerc)

$$\rho_m \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \rho_q \mathbf{E} + \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathcal{P}$$

Cf. the **Euler equation** of fluid dynamics + electric force + magnetic force (**Ampère's force**)

- plasma has been assumed fully ionized → the collision terms sum to 0
- electric force typically negligible, but the $\mathbf{J} \times \mathbf{B}$ force is essential
- non-linear terms make solutions difficult

$$\mathbf{V} \cdot \nabla \mathbf{V} \quad \mathbf{J} \times \mathbf{B} \propto B^2$$

The current transfer eq. ("**generalized Ohm's law**") is much more complicated. Multiply by q_{α}/m_{α} and sum over α (here only e and i, i.e., two-fluid case):

$$\begin{aligned} \frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot (\mathbf{V} \mathbf{J} + \mathbf{J} \mathbf{V} - \mathbf{V} \mathbf{V} \rho_q) &= \sum_{\alpha} \frac{n_{\alpha} q_{\alpha}^2}{m_{\alpha}} \mathbf{E} \\ &+ \left(\frac{e^2}{m_e} + \frac{e^2}{m_i} \right) \frac{\rho_m \mathbf{V} \times \mathbf{B}}{m_e + m_i} - \left(\frac{em_i}{m_e} - \frac{em_e}{m_i} \right) \frac{\mathbf{J} \times \mathbf{B}}{m_e + m_i} \\ &- \frac{e}{m_e} \nabla \cdot \left(\mathcal{P}_i^{CM} \frac{m_e}{m_i} - \mathcal{P}_e^{CM} \right) + \sum_{\alpha} \int q_{\alpha} \mathbf{v} \left(\frac{\partial f_{\alpha}}{\partial t} \right)_c d^3v. \end{aligned}$$

To make Ohm's law more transparent, let's again make some approximations:

Introduce the **average collision frequency** ν as:

$$\sum_{\alpha} \int q_{\alpha} \mathbf{v} \left(\frac{\partial f_{\alpha}}{\partial t} \right)_c d^3v = -\nu \mathbf{J}$$

This makes sense: Leaving out all derivatives and the magnetic field and recalling $\sigma = ne^2/\nu m_e$ we get the simple Ohm's law $\mathbf{J} = \sigma \mathbf{E}$

Some of the terms in Ohm's law are smaller than others $\propto m_e/m_i$

Typical approximations:

- In macroscopic considerations the quasi-neutrality implies $\nabla \cdot \mathbf{J} = 0$.
- When considering small perturbations to equilibrium, second order quantities \mathbf{VJ} , \mathbf{JV} , \mathbf{VV} can be omitted

$$\Rightarrow \underbrace{\mathbf{E} + \mathbf{V} \times \mathbf{B}}_{\text{MHD Ohm's law}} = \frac{\mathbf{J}}{\sigma} + \underbrace{\frac{1}{ne} \mathbf{J} \times \mathbf{B}}_{\text{Hall term}} - \underbrace{\frac{1}{ne} \nabla \cdot \mathcal{P}_e + \frac{m_e}{ne^2} \frac{\partial \mathbf{J}}{\partial t}}_{\text{neglected in basic MHD due to}}$$

- neglected in basic MHD due to
- slow temporal variations
 - weak spatial gradients

Case $\sigma \rightarrow \infty$:

Ideal MHD: $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$

Energy transport in MHD

Also the energy transport equation can be cast in the conservation form

$$\frac{\partial}{\partial t} \left[\underbrace{\frac{\rho_m V^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}}_{\text{temporal change of energy}} \right] = -\nabla \cdot \underbrace{\mathbf{H}}_{\text{divergence of energy (heat) flux}}$$

"Poynting theorem" of MHD

$$\mathbf{H} = \left(\frac{\rho_m V^2}{2} + \frac{\gamma p}{\gamma - 1} + \frac{B^2}{\mu_0} \right) \mathbf{V} - \frac{\mathbf{B}}{\mu_0} \left(\mathbf{V} + \frac{\mathbf{J}}{ne} \right) \cdot \mathbf{B} + \frac{\mathbf{J} \times \mathbf{B}}{\sigma \mu_0} + \frac{\mathbf{JB}^2}{\mu_0 ne} + \frac{m_e \mathbf{B}}{\mu_0 ne^2} \times \frac{\partial \mathbf{J}}{\partial t}$$

In the case of adiabatic ideal MHD, the energy equation can be written as

$$\frac{\partial}{\partial t} \left(\frac{\rho_m V^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{\rho_m V^2}{2} \mathbf{V} + \frac{\gamma p}{\gamma - 1} \mathbf{V} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) = 0$$

Equations of state

Determining the appropriate eq of state is non-trivial. In a real plasma the different particle populations (e, i) can have different thermodynamical behaviour

Assume **isotropic** pressure: $\mathcal{P} = p\mathcal{I}$

In **isothermal** processes $T = T_0 = \text{const.}$: $p = nk_B T_0$

This is valid if the system has time to thermalize with an external heat bath fast compared to time scale of the process.

The opposite limit is called **adiabatic**. In that case the system does not exchange energy with its surroundings.

Write the RHS of energy eq. (prev. slide) to zero and use $d/dt = \partial/\partial t + \mathbf{V} \cdot \nabla$

$$\Rightarrow \frac{3}{2}n \frac{dT}{dt} = T \frac{dn}{dt}$$

$$\Rightarrow T = T_0 \left(\frac{n}{n_0}\right)^{\gamma-1} ; p = p_0 \left(\frac{n}{n_0}\right)^{\gamma}$$

γ is the **polytropic index**

specific heats in const. p & V

- adiabatic: $\gamma = c_p/c_v = 5/3$
 - isothermal: $\gamma = 1$
 - isobaric: $\gamma = 0$
 - isometric: $\gamma = \infty$
- for 3D monoatomic gas!

MHD equations for isotropic pressure

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0$$

$$\rho_m \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} + \nabla p - \mathbf{J} \times \mathbf{B} = 0$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{J}/\sigma$$

$$p = p_0 \left(\frac{n}{n_0}\right)^{\gamma}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

closure by equation of state

displacement current neglected

Take scalar product with \mathbf{V}

and assume ideal MHD ($\sigma = \infty$) \Rightarrow

$$\frac{\partial}{\partial t} \left(\frac{\rho_m V^2}{2} + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{\rho_m V^2}{2} \mathbf{V} + \frac{\gamma p}{\gamma-1} \mathbf{V} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) = 0$$

Double adiabatic (CGL) theory

(anisotropic MHD is tricky)

Start from the ideal MHD

$$\begin{aligned}\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) &= 0 \\ \rho_m \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} + \nabla \cdot \mathcal{P} - \mathbf{J} \times \mathbf{B} &= 0 \\ \mathbf{E} + \mathbf{V} \times \mathbf{B} &= 0\end{aligned}$$

but let the pressure tensor be diagonal and gyrotropic

$$\mathcal{P} = \begin{pmatrix} p_{\perp} & 0 & 0 \\ 0 & p_{\perp} & 0 \\ 0 & 0 & p_{\parallel} \end{pmatrix}$$

At individual particle level the dynamics perpendicular and parallel to \mathbf{B} are different

Assume further that

$$\begin{aligned}p_{\parallel} &= nk_B T_{\parallel} \\ p_{\perp} &= nk_B T_{\perp}\end{aligned}$$

Consider the adiabatic case and recall from thermodynamics that

$$\gamma = (d+2)/d \quad \text{where } d \text{ is the dimension (e.g. 3D): } \gamma = 5/3$$

Now the parallel space is 1D ($\gamma = 3$) and the perpendicular 2D ($\gamma = 2$)

But this reasoning is **WRONG**, because the parallel and perpendicular dynamics are coupled to each other (e.g., the mirror force)!

Assume that the motion of individual particles is adiabatic, then $\mu = W_{\perp}/B$ & $\langle \mu \rangle = k_B T_{\perp}/B = p_{\perp}/(nB)$ are conserved $\Rightarrow \frac{d}{dt} \left(\frac{p_{\perp}}{nB} \right) = 0$

The parallel motion is more complicated. With some simplifying approximations (e.g., neglecting the heat flux) we get

$$\frac{d}{dt} \left(\frac{p_{\perp}^2 p_{\parallel}}{n^5} \right) = \frac{d}{dt} \left(\frac{p_{\parallel} B^2}{n^3} \right) = 0$$

These equations are known as **double adiabatic theory** (or CGL-theory after Chew, Goldberger and Low)

Writing the pressure tensor as $\mathcal{P} = p_{\perp} \mathbf{I} + (p_{\parallel} - p_{\perp}) \mathbf{b}\mathbf{b}$

the momentum equation is split into two parts

$$\begin{aligned}\rho_m \left(\frac{d\mathbf{V}}{dt} \right)_{\perp} + \nabla_{\perp} \left(p_{\perp} + \frac{B^2}{2\mu_0} \right) - \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{\mu_0} \left(\frac{p_{\perp} - p_{\parallel}}{B^2/\mu_0} + 1 \right) &= 0 \\ \rho_m \left(\frac{d\mathbf{V}}{dt} \right)_{\parallel} + \nabla_{\parallel} p_{\parallel} + (p_{\perp} - p_{\parallel}) \left(\frac{\nabla B}{B} \right)_{\parallel} &= 0.\end{aligned}$$

In the CGL theory $T_{\perp} \propto B$; $T_{\parallel} \propto (n/B)^2$

But this is not true, e.g., above the auroral zone

The problem is at least partly in the neglect of heat transfer:
Particles leaking out from the magnetic mirror carry heat with them
away from the system.

Actually the CGL theory does not seem to be very useful in practice.

Note that in the CGL theory there are no constant polytropic indices
Assuming $p_{\parallel} \propto n^{\gamma_{\parallel}}$ and $p_{\perp} \propto n^{\gamma_{\perp}}$ we can derive expressions

$$\gamma_{\perp} = 1 + \frac{\ln(B/B_0)}{\ln(n/n_0)}$$
$$\gamma_{\parallel} = 3 - 2 \frac{\ln(B/B_0)}{\ln(n/n_0)}$$

These are related to each other as $\gamma_{\parallel} + 2\gamma_{\perp} = 5$
but both of them vary in inhomogeneous plasma

Some remarks

- Collective effects of free charges determine the behavior of the plasma as an electromagnetic medium
- Collisionless space plasmas may have properties, that are difficult to cast in the form of constitutive relationships
 - e.g., above auroral arcs there is a large-scale current-voltage relationship but no local Ohm's law
- Plasma behaves nonlinearly
 - Vlasov equation is nonlinear, magnetohydrostatic equilibrium is nonlinear, etc.
 - linearizations are often useful, e.g., to find the normal modes of plasma oscillations, but the observable plasma oscillations are either damped or grow to nonlinear level leading to instabilities
- Plasma is turbulent
 - plasma turbulence is an even more complicated issue than ordinary fluid turbulence
- Plasma systems often exhibit chaotic behavior
 - concepts of chaos, such as self-organized criticality, intermittence, renormalization groups, etc., are important in theoretical plasma physics.