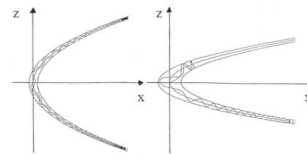
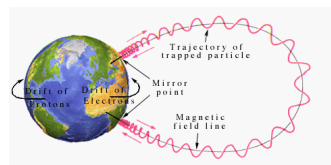


## Single particle motion

The task is to determine individual particle orbits from the equation of motion in some of the most important magnetic field configuration, as well in temporally varying electric fields (wave)

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{F}_{non-EM}$$

Assume first nonrelativistic motion  $\gamma = 1$ ,  $\mathbf{p} = m\mathbf{v}$



## Guiding centre approximation

Equation of motion of charged particles is

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{F}_{non-EM} \quad (\text{assume, for the time being, nonrelativistic motion; } \gamma = 1 \text{ and } \mathbf{p} = m\mathbf{v})$$

Consider the case  $\mathbf{E} = 0$  and  $\mathbf{B} = \text{const}$  (neglect the non-EM forces)

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B})$$

$\omega_c = \frac{qB}{m}$

cyclotron frequency  
gyro frequency  
Larmor frequency

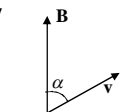
The radius of the circle is (Larmor radius)  $r_L = \frac{v_{\perp}}{|\omega_c|} = \frac{mv_{\perp}}{|q|B}$  ;  $v_{\perp} = \sqrt{v_x^2 + v_y^2}$

The gyro period (cyclotron period, Larmor time) is:  $\tau_L = \frac{2\pi}{|\omega_c|}$

The pitch angle ( $\alpha$ ) of the helical path is defined by

$$\tan \alpha = v_{\perp} / v_{\parallel}$$

$$\alpha = \arcsin(v_{\perp} / v) = \arccos(v_{\parallel} / v)$$



The frame of reference where  $v_{\parallel} = 0$  : **Guiding centre system (GCS)**

Decomposition of the motion to the motion of the guiding centre and to the gyro motion is called the **guiding centre approximation**

In the GCS the charge causes an electric current:  $I = q / \tau_L$

The **magnetic moment** associated to the circular loop is

$$\mu = I \pi r_L^2 = \frac{1}{2} \frac{q^2 r_L^2 B}{m} = \frac{1}{2} \frac{m v_{\perp}^2}{B} = \frac{W_{\perp}}{B}$$

or, in the vector form  $\mu = \frac{1}{2} q \mathbf{r}_L \times \mathbf{v}_{\perp}$

Clearly:  $\mu$  is always opposite to  $\mathbf{B}$  ( $r_L$  depends on the sign of  $q$ )

Thus plasma can be considered a **diamagnetic** medium:

Placing a large number of charged particles to external magnetic field reduces the field.

## E x B drift

Let  $\mathbf{E} = \text{const}$  and  $\mathbf{B} = \text{const}$

The eq. of motion along  $\mathbf{B}$  is  $m \dot{v}_{\parallel} = q E_{\parallel}$

- constant acceleration parallel/antiparallel to  $\mathbf{B}$
- very rapid cancellation of large-scale  $\mathbf{E}_{\parallel}$  in plasma!

The perpendicular components of the eq. of motion are

$$\begin{aligned} \dot{v}_x &= \omega_c v_y + \frac{q}{m} E_x & \ddot{v}_x &= -\omega_c^2 v_x \\ \dot{v}_y &= -\omega_c v_x & \ddot{v}_y &= -\omega_c^2 \left( v_y + \frac{E_x}{B} \right) \end{aligned}$$

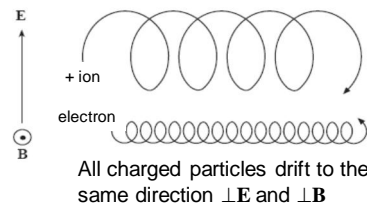
Substitution  $v'_y = v_y + E_x/B$  leads again to gyro motion but now the GC drifts in the  $y$ -direction with speed  $E_x/B$

In vector form:  $\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$

Same as making Lorentz transformation to GCS:

$$\mathbf{E}' = \mathbf{E} + \mathbf{w} \times \mathbf{B} \quad (\text{non-relat. } \gamma = 1)$$

where  $\mathbf{E}' = 0$ ,  $\mathbf{E} = -\mathbf{w} \times \mathbf{B}$



## Other non-magnetic drifts

Write the perpendicular eq. of motion in the form  $\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B}) + \frac{\mathbf{F}_\perp}{m}$

Assume that  $\mathbf{F}_\perp$  gives rise to a drift  $\mathbf{v}_D$  and transform  $\mathbf{v}_\perp = \mathbf{v}'_\perp + \mathbf{v}_D$

$$\Rightarrow \frac{d\mathbf{v}'_\perp}{dt} = \frac{q}{m}(\mathbf{v}'_\perp \times \mathbf{B}) + \frac{q}{m}(\mathbf{v}_D \times \mathbf{B}) + \frac{\mathbf{F}_\perp}{m}$$

In GCS the last two terms must sum to 0  $\Rightarrow \mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}$  (\*)

This requires  $F/qB \ll c$ . If  $F > qcB$ , the GC approximation cannot be used!

Inserting  $\mathbf{F}_\perp = q\mathbf{E}$  into (\*) we get the ExB-drift

$\mathbf{F}_\perp = mg$  gives the **gravitational drift**  $\mathbf{v}_g = \frac{m\mathbf{g} \times \mathbf{B}}{qB^2} \propto \frac{m}{q}$  separates charges  $\rightarrow$  current

## Drifts due to magnetic inhomogeneities

Assume static but inhomogeneous magnetic field. Guiding centre approximation is useful if the spatial and temporal gradients of  $\mathbf{B}$  are small as compared to the gyro motion:

$$|\partial\mathbf{B}/\partial t| \ll \omega_c B \quad ; \quad |\nabla B|_\perp \ll B/r_L \quad ; \quad |\nabla B|_\parallel \ll (\omega_c/v_\parallel)B$$

Expand the field as a Taylor series about the GC:

$$\mathbf{B}(\mathbf{r}) \simeq \mathbf{B}_0 + \mathbf{r} \cdot (\nabla\mathbf{B})_0 + \dots \quad \text{where } \mathbf{B}_0 \text{ is the field at GC}$$

Move to the frame of reference, where  $v_\parallel = 0$

Write  $\mathbf{v} = \mathbf{v}_0 + \mathbf{u}$  solution for const  $\mathbf{B}$  small correction

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}_0) + \frac{q}{m}(\mathbf{v} \times [\mathbf{r} \cdot (\nabla\mathbf{B})_0]) + \dots$$

this contains a second order term  $\mathbf{u} \times [\mathbf{r} \cdot (\nabla\mathbf{B})_0]$

Small perturbation:  $\mathbf{r} \approx r_L$  Thus the the first order equation of motion is

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}_0) + \frac{q}{m}(\mathbf{v}_0 \times [r_L \cdot (\nabla\mathbf{B})_0])$$

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}_0) + \frac{q}{m}(\mathbf{v}_0 \times [\mathbf{r}_L \cdot (\nabla \mathbf{B})_0])$$

Recall the similarity with the zero order equation

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B}) + \frac{\mathbf{F}_\perp}{m} \quad \text{from which we found} \quad \mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}$$

$$\text{Now } \mathbf{F} = q(\mathbf{v}_0 \times [\mathbf{r}_L \cdot (\nabla \mathbf{B})_0])$$

It is easiest to consider gradients and curvature of  $\mathbf{B}$  separately.

We start with straight field lines, when it is natural to use cylindrical coordinates around  $\mathbf{B}_0$ :

$$\mathbf{e}_z \parallel \mathbf{B}_0, \quad \mathbf{e}_\phi \parallel \mathbf{v}_0, \quad \mathbf{B} = B_r \mathbf{e}_r + B_\phi \mathbf{e}_\phi + B_z \mathbf{e}_z$$

For GC motion we can average over the gyro orbit:  $\langle \rangle$

$$\text{An easy exercise gives } \mathbf{F} = \left\langle -q\mathbf{v}_0 \times r_L \left( \frac{\partial \mathbf{B}}{\partial r} \right)_0 \right\rangle$$

$$\text{Omit the subscript } 0 \text{ and write } \begin{cases} \mathbf{F}_\parallel = \left\langle q\mathbf{v} \times r_L \left( \frac{\partial B_r}{\partial r} \right) \right\rangle \\ \mathbf{F}_\perp = \left\langle -q\mathbf{v} \times r_L \left( \frac{\partial B_z}{\partial r} \right) \mathbf{e}_z \right\rangle \end{cases}$$

To find the parallel force note that  $\mathbf{v} \times \mathbf{r}_L = (2\mu/q) \mathbf{e}_z$

$$\Rightarrow \mathbf{F}_\parallel = 2\mu \left\langle \frac{\partial B_r}{\partial r} \right\rangle \mathbf{e}_z = -\mu \left( \frac{\partial B_z}{\partial z} \right) \mathbf{e}_z = -\mu \nabla_\parallel B$$

To calculate the perpendicular force we select the  $xy$  plane as the plane of gyro motion

$$\mathbf{v} \times \mathbf{e}_z = -\frac{r_L}{r_L} v$$

$$\partial/\partial r = \cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y}$$

$$\mathbf{r}_L = r_L (\cos \phi \mathbf{e}_x + \sin \phi \mathbf{e}_y)$$

When calculating the averages, recall:

$$\langle \cos^2 \phi \rangle = \langle \sin^2 \phi \rangle = 1/2 \quad \& \quad \langle \sin \phi \cos \phi \rangle = 0$$

$$\text{Another simple exercise gives us } \mathbf{F}_\perp = -\frac{qvr_L}{2} \left\langle \frac{\partial B_z}{\partial x} \mathbf{e}_x + \frac{\partial B_z}{\partial y} \mathbf{e}_y \right\rangle$$

$$\text{Because } \nabla_\perp = \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} \quad \text{we can write } \mathbf{F}_\perp = -\mu \nabla_\perp B$$

and the total force on the GC is  $\mathbf{F} = -\mu \nabla B$

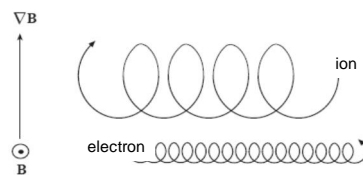
The parallel force gives acceleration along  $\mathbf{B}$

$$\frac{d\mathbf{v}_\parallel}{dt} = -\frac{\mu}{m} \nabla_\parallel B$$

$$\text{and from the equation } \mathbf{v}_D = \frac{\mathbf{F}_\perp \times \mathbf{B}}{qB^2}$$

we get the gradient drift velocity

$$\mathbf{v}_G = \frac{\mu}{qB^2} \mathbf{B} \times (\nabla B) = \frac{W_\perp}{qB^3} \mathbf{B} \times (\nabla B)$$



The reference frame where  $v_{\parallel} = 0$  is not inertial, because  $dv_{\parallel}/dt \neq 0$

In a curved  $\mathbf{B}$  also the orbit of GC is curved

Denote the GC velocity by  $w$  ( $w_{\parallel} \neq v_{\parallel}$ ).

Now we move to the frame co-moving with the GC

Choose an orthonormal basis  $\{e_i\}$  ( $i = 1, 2, 3$ ) where  $e_3 \parallel v_{\parallel} \parallel \mathbf{B}$

$\mathbf{v} = \sum v_i e_i$ , and the basis may rotate when its origin moves with the GC

$$\frac{d\mathbf{v}}{dt} = \sum_i \left( \frac{dv_i}{dt} e_i + v_i \frac{de_i}{dt} \right) = \sum_i \left( \frac{dv_i}{dt} e_i + v_i \underbrace{(w_{\parallel} \cdot \nabla) e_i}_{\text{effect of curvature}} \right)$$

To find the centrifugal force over one gyro period, take again the average effect of curvature  
→ centrifugal force

$$\mathbf{F}_C = - \left\langle m \sum_i v_i (w_{\parallel} \cdot \nabla) e_i \right\rangle \quad \text{Assuming weak curvature}$$

$(w_{\parallel} \cdot \nabla) e_i$  is constant during one rotation

$v_1$  &  $v_2$  make sinusoidal oscillation  $\Rightarrow \langle v_1 e_1 \rangle = \langle v_2 e_2 \rangle = 0$

During on gyroperiod  $v_{\parallel} \approx w_{\parallel} \Rightarrow \mathbf{F}_C = -mw_{\parallel}^2 (e_3 \cdot \nabla) e_3$

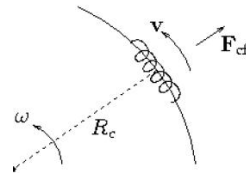
An exercise in differential geometry yields  $(e_3 \cdot \nabla) e_3 = \mathbf{R}_C / R_C^2$   
where  $\mathbf{R}_C$  is the inward pointing radius of curvature

$$\Rightarrow \mathbf{F}_C = -mw_{\parallel}^2 \frac{\mathbf{R}_C}{R_C^2}$$

$$\mathbf{B} = B e_3 \Rightarrow (e_3 \cdot \nabla) e_3 = (\mathbf{B} \cdot \nabla)_{\perp} / B^2$$

$$\text{Using again } \mathbf{v}_D = \frac{\mathbf{F}_{\perp} \times \mathbf{B}}{qB^2}$$

$$\mathbf{v}_C = \frac{-mw_{\parallel}^2}{qB^2} \frac{\mathbf{R}_C \times \mathbf{B}}{R_C^2} = \frac{mw_{\parallel}^2}{qB^4} \mathbf{B} \times (\mathbf{B} \cdot \nabla) \mathbf{B}$$



Approximating again  $v_{\parallel} \approx w_{\parallel}$

we can write this in terms of energy  $W_{\parallel} \approx (1/2)mw_{\parallel}^2$

If there are no local currents in plasma, i.e.,  $\nabla \times \mathbf{B} = 0$ ,

the curvature drift reduces to  $\mathbf{v}_C = \frac{2W_{\parallel}}{qB^3} \mathbf{B} \times \nabla \mathbf{B}$

and we can combine the gradient and curvature drifts

$$\mathbf{v}_{GC} = \frac{W_{\perp} + 2W_{\parallel}}{qB^3} \mathbf{B} \times \nabla \mathbf{B} = \frac{W}{qBR_C} (1 + \cos^2 \alpha) \mathbf{n} \times \mathbf{t} \quad \left| \begin{array}{l} \mathbf{t} \parallel \mathbf{B} \text{ \& \ } \mathbf{n} \parallel \mathbf{R}_C \\ \text{are unit vectors} \end{array} \right.$$

These are straightforward to modify for relativistic motion by substitution  $\gamma m \rightarrow m$

## Adiabatic invariants

Symmetry principles: periodic motion ↔ conserved quantity  
 symmetry ↔ conservation law

What if the motion is **almost** periodic?

Hamiltonian mechanics:

Let  $q$  &  $p$  be canonical variables and the motion almost periodical ⇒

$$I = \oint p dq \quad \text{is constant, called **adiabatic invariant**}$$

Example: Consider a charged particle in Larmor motion.

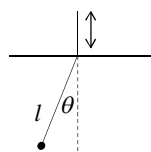
Assume that the  $\mathbf{B}$  does not change much during within one circle.

The canonical coordinate is  $\mathbf{r}_L$  and the canonical momentum  $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$

$$\begin{aligned} I &= \oint \mathbf{p}_\perp \cdot d\mathbf{r}_L = \oint m\mathbf{v}_\perp \cdot d\mathbf{r}_L + q \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} && \text{charge!} \\ &= \int_0^{2\pi r_L} m v_\perp dl + q \int_S \mathbf{B} \cdot d\mathbf{S} \\ &= 2\pi m v_\perp r_L - |q| B \pi r_L^2 = \frac{2\pi m}{|q|} \mu \end{aligned}$$

i.e., the magnetic moment is an adiabatic invariant

## Example: Lorentz-Einstein pendulum



Assume that the length of the pendulum is changed slowly (as compared to  $\omega$ ) ⇒

$$\omega = \sqrt{\frac{g}{l}} \quad \text{changes slowly}$$

Now the energy per mass unit  $W = \frac{1}{2}l^2\dot{\theta}^2 + \frac{1}{2}gl\theta^2$  is not conserved.

Lorentz asked Einstein at a conference in 1911:

What is the conserved quantity in this case?

Einstein:  $W/\omega$ .

This was a relevant question when quantum mechanics was being developed. Recall that

$$W/\omega = n\hbar = \text{const}$$

Relation to the conservation of magnetic moment:

$$\mu = \frac{W_\perp}{B} = \frac{q}{m} \frac{W_\perp}{\omega_c}$$

## First adiabatic invariant

In plasma physics  $\mu$  is called **the first adiabatic invariant**.  
 Let's show its conservation by direct calculation in the case of  
 a static weakly inhomogeneous  $\mathbf{B}$  (GC approximation)

The system is conservative, i.e., total energy is conserved:

$$W = W_{\parallel} + W_{\perp} = \text{const} \quad \Rightarrow \quad \frac{dW_{\parallel}}{dt} + \frac{dW_{\perp}}{dt} = 0$$

$$W_{\perp} = \mu B \quad \Rightarrow \quad \frac{dW_{\perp}}{dt} = \mu \frac{dB}{dt} + \frac{d\mu}{dt} B$$

Change in parallel energy:  $\frac{dW_{\parallel}}{dt} = -\mu \nabla_{\parallel} B = -\mu \frac{dB}{ds}$   
 along the path of the GC  
 multiply LHS by  $v_{\parallel}$  and RHS by  $ds/dt \Rightarrow$

$$\frac{dW_{\parallel}}{dt} = -\mu \frac{dB}{dt} \quad \text{and thus} \quad \frac{dW_{\parallel}}{dt} + \frac{dW_{\perp}}{dt} = B \frac{d\mu}{dt} = 0$$

i.e.,  $\mu$  is constant in the guiding centre approximation

Assume next that  $\mathbf{B}$  is time-dependent but changes slowly:  $\partial/\partial t \ll \omega_c$

Now the electric field must be taken into account  $\partial\mathbf{B}/\partial t = -\nabla \times \mathbf{E}$

The electric field changes the perpendicular energy

$$\frac{dW_{\perp}}{dt} = q(\mathbf{E} \cdot \mathbf{v}_{\perp}) \quad \text{which during one gyro period yields the work}$$

$$\Delta W_{\perp} = q \int_0^{2\pi/\omega_c} \mathbf{E} \cdot \mathbf{v}_{\perp} dt$$

As the field changes slowly, replace the time integral by contour integral

$$\Delta W_{\perp} = q \oint_C \mathbf{E} \cdot d\mathbf{l} = q \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -q \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

↑ Stokes
↑ Faraday

For small changes of the field

$$\partial\mathbf{B}/\partial t \rightarrow \omega_c \Delta B / 2\pi \quad \Rightarrow \quad \Delta W_{\perp} = \frac{1}{2} q \omega_c r_L^2 \Delta B = \mu \Delta B$$

Because  $\Delta W_{\perp} = \mu \Delta B + B \Delta \mu$   $\mu$  must again remain constant

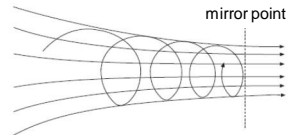
Note that also the flux enclosed by the gyro orbit  $\Phi = B \pi r_L^2 = \frac{2\pi m}{q^2} \mu$   
 is constant

## Magnetic mirror

In guiding centre approximation both  $W$  and  $\mu$  are conserved.  
 If  $B$  increases slowly,  $W_{\perp}$  increases slowly, thus  $W_{\parallel}$  decreases.  
 What happens when  $W_{\parallel} \rightarrow 0$  ?

$$\mu = \frac{mv^2 \sin^2 \alpha}{2B} \quad \text{As } \mu \text{ and } v^2 \text{ are conserved, } \alpha \text{ and } B \text{ are related through}$$

$$\frac{\sin^2 \alpha_1}{\sin^2 \alpha_2} = \frac{B_1}{B_2}$$



When  $\alpha \rightarrow \pi/2$ , the force  $\mathbf{F} = -\mu \nabla_{\parallel} B$

on the GC turns the charge back

(**mirror force**) and the **mirror field**  $B_m$

for a charge that at  $B_0$  has

the pitch-angle  $\alpha$  is given by  $\sin^2 \alpha_0 = B_0/B_m$

Also parallel electric field and/or  
 gravitation may need to be considered

$$m \frac{dv_{\parallel}}{dt} = qE_{\parallel} + mg_{\parallel} - \mu \nabla_{\parallel} B$$

If the non-magnetic forces can  
 be derived from a potential  $U(s)$ ,

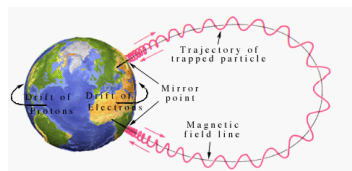
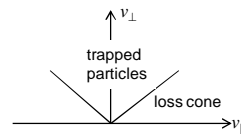
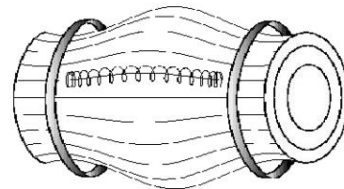
$$m \frac{dv_{\parallel}}{dt} = -\frac{\partial}{\partial s} [U(s) + \mu B(s)]$$

## Magnetic bottle

A simple **magnetic bottle** consists  
 of two mirrors facing each other.  
 A charged particle is **trapped**  
 in the bottle if

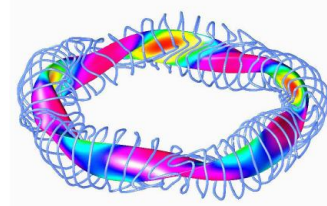
$$\arcsin \sqrt{\frac{B_0}{B_m}} \leq \alpha_0 \leq 180 - \arcsin \sqrt{\frac{B_0}{B_m}}$$

Otherwise it is said to be in the **loss-cone**  
 and escape at the end of the bottle



The dipole field of the Earth is  
 a large magnetic bottle

There are much more complicated  
 trapping configurations, e.g. stellarator:



## The second adiabatic invariant

Let  $q$  &  $p$  be canonical variables and the motion almost periodical  $\Rightarrow$

$$I = \oint p dq \quad \text{is constant, called } \text{adiabatic invariant}$$

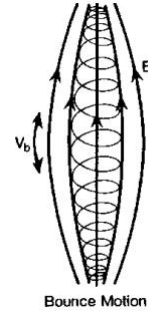
In plasma physics  $\mu$  is called the **first adiabatic invariant**.

Consider the bounce period of a charge in a magnetic bottle

$$\tau_b = 2 \int_{s_m}^{s'_m} \frac{ds}{v_{\parallel}(s)} = \frac{2}{v} \int_{s_m}^{s'_m} \frac{ds}{(1 - B(s)/B_m)^{1/2}}$$

If  $\tau_b \frac{dB/dt}{B} \ll 1$  then  $J = \oint p_{\parallel} ds$

is constant (**second adiabatic invariant**)



This, of course requires that  $\tau_b \gg \tau_L$

A direct proof of the invariance of  $J$  is a rather tedious task

## Betatron acceleration

Let  $T$  be the kinetic energy of the particle in a time-dependent  $\mathbf{B}$

Write the time derivative in a moving frame as  $d/dt = \partial/\partial t + \mathbf{w} \cdot \nabla$

In GCS:

$$\frac{dT_{GCS}}{dt} = \mu \frac{dB}{dt} = \mu \left( \frac{\partial B}{\partial t} + \mathbf{w}_{\perp} \cdot \nabla_{\perp} B + w_{\parallel} \frac{\partial B}{\partial s} \right)$$

In the reference frame of the observer (OFR)

$$\frac{dT_{OFR}}{dt} = \frac{dT_{GCS}}{dt} + \frac{d}{dt} \left( \frac{1}{2} m w_{\parallel}^2 \right) + \frac{d}{dt} \left( \frac{1}{2} m w_{\perp}^2 \right)$$

Do a little algebra  $\Rightarrow$  betatron acceleration:

$$\frac{dT_{OFR}}{dt} = \mu \frac{\partial B}{\partial t} + q \mathbf{w} \cdot \mathbf{E}$$

Two effects:

- Field-aligned acceleration, if  $E_{\parallel} \neq 0$ .

- Drift betatron:

Particle drifts toward increasing  $B$ :  $B_2 > B_1$

Conservation of  $\mu \Rightarrow$

$$\frac{W_{\perp 2}}{W_{\perp 1}} = \frac{B_2}{B_1} \quad \text{thus} \quad W_{\perp 2} > W_{\perp 1}$$

Increasing  $B$  at the position of GC: "gyro betatron"

## Fermi acceleration of cosmic rays

Fermi proposed the drift betatron acceleration as a mechanism to accelerate cosmic rays to very large energies in 1949 in the following form:

Let the particle move in a mirror field configuration where the mirror points move toward each other.

Assume that

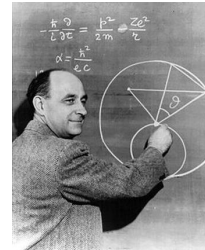
$$J = \oint p_{\parallel} ds$$

is conserved.

Now  $\oint ds$  decreases. To compensate this  $v_{\parallel}$  and thus  $W_{\parallel}$  must increase.

Compare this to the acceleration of a tennis ball hit by a racket!

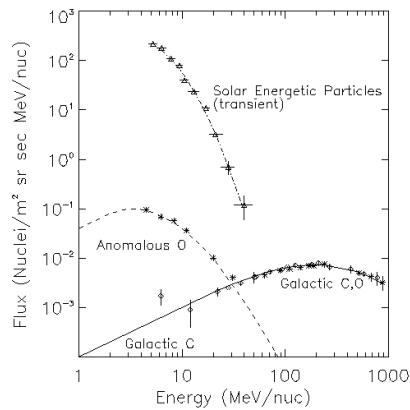
The modern version of Fermi acceleration is called **diffusive shock acceleration** where shock waves emerging from supernova explosions are responsible for the acceleration. It does not conserve  $\mu$  nor  $J$ .



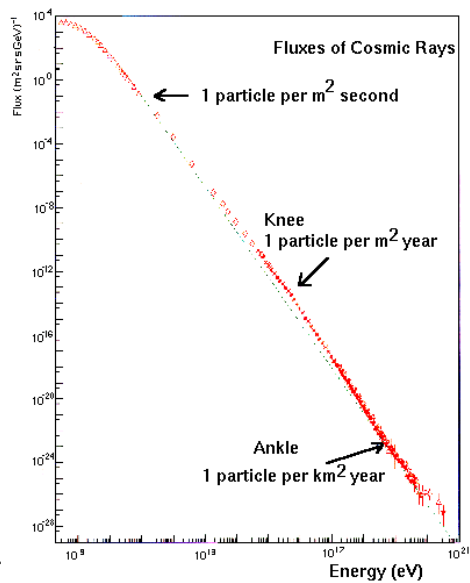
Enrico Fermi

### Cosmic Rays

- galactic: > 100 MeV
- solar: < 1 GeV
- Anomalous: around 10 MeV

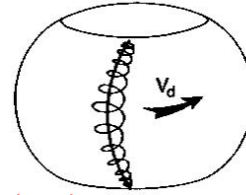


### Spectrum of galactic cosmic rays



## Third adiabatic invariant

If the perpendicular drift of the GC is nearly periodic (e.g. in a dipole field), the magnetic flux through the GC orbit



$\Phi = \oint \mathbf{A} \cdot d\mathbf{x}$  is conserved. This is the **third adiabatic invariant**.

## Summary

Invariant	Velocity	Time scale	Validity
magnetic moment $\mu$	gyro motion $\mathbf{v}_\perp$	gyro period $\tau_L = 2\pi/\omega_c$	$\tau \gg \tau_L$
longitudinal invariant $J$	longitudinal velocity of GC $\mathbf{w}_\parallel$	bounce period $\tau_b$	$\tau \gg \tau_b \gg \tau_L$ and $\mu$ const
flux invariant $\Phi$	perpendicular velocity of GC $\mathbf{w}_\perp$	drift period $\tau_d$	$\tau \gg \tau_d \gg \tau_b \gg \tau_L$ and $\mu$ and $J$ const

There is a specific energization mechanism for each invariant

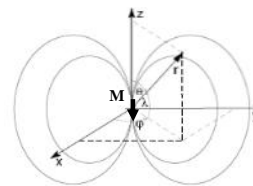
- $\mu$ :  $W_\perp$  changed by changing the Larmor radius (i.e.,  $|B|$ )
- $J$ :  $W_\parallel$  changed by stretching or shortening the magnetic bottle
- $\Phi$ :  $W$  changed by compressing or expanding the drift surface

## Motion in a dipole field

The dipole field is familiar from basic electrodynamics.

We use the notations of geomagnetism:

- the **latitude**  $\lambda$ : zero at equator, increases towards north
- the **longitude**  $\phi$  increases towards east
- instead of **dipole moment**  $M_E$  ( $E$  for the Earth), we use  $k_0 = \mu_0 M_E / 4\pi$  (also often called dipole moment)



$$\begin{aligned}
 M_E &= 8 \times 10^{22} \text{ Am}^2 \\
 k_0 &= 8 \times 10^{15} \text{ Wb m} \quad (\text{SI: Wb} = \text{Tm}^2) \\
 &= 8 \times 10^{25} \text{ G cm}^3 \quad (\text{Gaussian units } G = 10^{-4} \text{ T}) \\
 &= 0,3 \text{ GR}_E^3 \quad (R_E \simeq 6370 \text{ km, Earth's radius})
 \end{aligned}$$

Earth's dipole moment points southward

As the dipole field is curl-free, it can be obtained as  $\nabla\Psi$ , where

$$\Psi = -\mathbf{k}_0 \cdot \nabla \frac{1}{r} = -k_0 \frac{\sin \lambda}{r^2}$$

⇒

$$\mathbf{B} = \frac{1}{r^3} [3(\mathbf{k}_0 \cdot \mathbf{e}_r)\mathbf{e}_r - \mathbf{k}_0]$$

components of  $\mathbf{B}$ :

$$\begin{cases}
 B_r = -\frac{2k_0}{r^3} \sin \lambda \\
 B_\lambda = \frac{k_0}{r^3} \cos \lambda \\
 B_\phi = 0, \quad \leftarrow (\text{azimuthal symmetry})
 \end{cases}$$

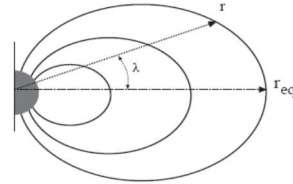
magnitude of  $\mathbf{B}$ :

$$B = \frac{k_0}{r^3} (1 + 3 \sin^2 \lambda)^{1/2}$$

The field lines are found from

the equation  $\frac{dr}{B_r} = \frac{rd\lambda}{B_\lambda}$

$\Rightarrow r = r_0 \cos^2 \lambda$



The **length of line element** is now

$$ds = (dr^2 + r^2 d\lambda^2)^{1/2} = r_0 \cos \lambda (1 + 3 \sin^2 \lambda)^{1/2} d\lambda$$

Note the important geometrical factor  $(1 + 3 \sin^2 \lambda)^{1/2} = (4 - 3 \cos^2 \lambda)^{1/2}$

Every field line is determined by two parameters:

(constant) longitude  $\phi_0$  and distance  $r_0$  where the line crosses the equator

Define the **L-parameter**:  $L = r_0/R_E$

For given  $L$  the field line crosses the surface at latitude  $\lambda_e = \arccos \frac{1}{\sqrt{L}}$

$B$  on a given field line as a function of latitude:

$$B(\lambda) = [B_r(\lambda)^2 + B_\lambda(\lambda)^2]^{1/2} = \frac{k_0 (1 + 3 \sin^2 \lambda)^{1/2}}{r_0^3 \cos^6 \lambda}$$

for the Earth:  $\frac{k_0}{r_0^3} = \frac{0.3}{L^3} \text{ G} = \frac{3 \times 10^{-5}}{L^3} \text{ T}$

In calculations of particle orbits we need the **curvature radius** of  $\mathbf{B}$ ,  $R_C$ .

$$1/R_C = |d^2 \mathbf{r} / ds^2| \Rightarrow R_C(\lambda) = \frac{r_0}{3} \cos \lambda \frac{(1 + 3 \sin^2 \lambda)^{3/2}}{2 - \cos^2 \lambda}$$

Guiding centre approximation is applicable in dipole field, if  $R_L \ll R_C$  (\*)

Expressed in terms of the **rigidity** of the particle  $mv_\perp / |q|$

$$r_L \left| \frac{\nabla_\perp B}{B} \right| = \frac{mv_\perp}{|q| R_C B} \propto \frac{mv_\perp}{|q| r_0 B}$$

In cosmic ray physics rigidity is oftend defined by  $R = pc/|q|$ , and thus  $[R] = \text{V}$

Condition (\*) reduces to  $\frac{mv_\perp}{|q|} \ll r_0 B$

The dipole field is a magnetic bottle. Let  $\lambda_m$  be the mirror latitude.

The pitch-angle at equator is determined by

$$\sin^2 \alpha_0 = \frac{B_0}{B(\lambda_m)} = \frac{\cos^6 \lambda_m}{(1 + 3 \sin^2 \lambda_m)^{1/2}}$$

The width of the loss cone at equator is given by

$$\sin^2 \alpha_{0l} = L^{-3} (4 - 3/L)^{-1/2} = (4L^6 - 3L^5)^{-1/2}$$

The particle leaks out if its  $\alpha_0 < \alpha_{0l}$

At Earth the particles are lost at an altitude of about 100 km (in the ionosphere)

Consider the bounce motion in the dipole field:

$$\begin{aligned}\tau_b &= 4 \int_0^{\lambda_m} \frac{ds}{v_{\parallel}} = 4 \int_0^{\lambda_m} \frac{ds}{d\lambda} \frac{d\lambda}{v_{\parallel}} \\ &= \frac{4r_0}{v} \int_0^{\lambda_m} \frac{\cos \lambda (1 + 3 \sin^2 \lambda)^{1/2}}{1 - \sin^2 \alpha_0 (1 + 3 \sin^2 \lambda)^{1/2} / \cos^6 \lambda} d\lambda \\ &= \frac{4r_0}{v} f(\alpha_0),\end{aligned}$$

Here the following relationships have been used:

$$\begin{aligned}v_{\parallel}(\lambda) &= v \cos \alpha = v [1 - \sin^2 \alpha]^{1/2} = v [1 - \sin^2 \alpha_0 B(\lambda)/B(0)]^{1/2} \\ B(\lambda) &= [B_r(\lambda)^2 + B_{\lambda}(\lambda)^2]^{1/2} = \frac{k_0 (1 + 3 \sin^2 \lambda)^{1/2}}{r_0^3 \cos^6 \lambda}\end{aligned}$$

For pitch-angles  $30^\circ \leq \alpha_0 \leq 90^\circ$ :  $f(\alpha_0) \approx 1,30 - 0,56 \sin^2 \alpha_0$

A useful order-of-magnitude estimate:  $\tau_b \approx \frac{4r_0}{v} \approx 0,085 \cdot \frac{L}{\beta} \text{ s}$  ;  $\beta = v/c$

In the Earth's dipole field 1 keV electrons bounce in seconds,  
1 keV protons in minutes.

As  $\nabla \times \mathbf{B} = 0$  in the dipole field, we can write the gradient and curvature drifts as

$$\begin{aligned}v_{GC} &= \frac{W}{qBR_C} (1 + \cos^2 \alpha(\lambda)) \\ &= \frac{3mv^2 r_0^2 \cos^5 \lambda (1 + \sin^2 \lambda)}{2qk_0 (1 + 3 \sin^2 \lambda)^2} \left[ 2 - \sin^2 \alpha_0 \frac{(1 + 3 \sin^2 \lambda)^{1/2}}{\cos^6 \lambda} \right]\end{aligned}$$

In case of the Earth: positive ions drift to the west, electrons to the east  
→ westward net current

Calculate the drift angular speed averaged over one bounce period

$$\begin{aligned}\langle \dot{\phi} \rangle &= \langle v_{GC} / r \cos \lambda \rangle \\ \langle \dot{\phi} \rangle &= \frac{4}{v\tau_b} \int_0^{\lambda_m} \frac{v_{GC}(\lambda) (1 + 3 \sin^2 \lambda)^{1/2}}{\cos^2 \lambda \cos \alpha(\lambda)} d\lambda \\ &\equiv \frac{3mv^2 r_0}{2qk_0} g(\alpha_0) = \frac{3mv^2 R_E L}{2qk_0} g(\alpha_0)\end{aligned}$$

$$\text{where } g(\alpha_0) = \frac{1}{f(\alpha_0)} \int_0^{\lambda_m} \frac{\cos^3 \lambda (1 + \sin^2 \lambda) [1 + \cos^2 \alpha(\lambda)]}{(1 + 3 \sin^2 \lambda)^{3/2} \cos \alpha(\lambda)} d\lambda$$

For  $30^\circ \leq \alpha_0 \leq 90^\circ$ :  $g(\alpha_0) \simeq 0,7 + 0,3 \sin(\alpha_0)$

For particles staying on equator ( $\alpha_0 = \pi/2$ )  $\langle \dot{\phi}_0 \rangle = \frac{3mv^2 R_E L}{2qk_0}$

For relativistic particles this is  $\langle \dot{\phi}_0 \rangle = \frac{3mc^2 R_E L}{2qk_0} \gamma \beta^2$   $\gamma = (1 - \beta^2)^{-1/2}$ .

Finally the **average drift time** is given by

$$\tau_d = \frac{2\pi}{|\langle \dot{\phi} \rangle|} = \frac{4\pi}{3} \frac{|q|k_0}{mc^2 R_E} \frac{1}{L\gamma\beta^2 g(\alpha_0)}$$

$$\approx 1,0 \cdot 10^4 \frac{m_e}{m} \frac{|q|}{e} \frac{1}{L\gamma\beta^2 g(\alpha_0)} \text{ s.}$$

Earth radiation belts (Van Allen belts):

Typical energies:

Inner belt protons: 0.1 MeV – 40 MeV

Outer belt electrons: keV – MeV

Typical drift periods

keV particles 100s of hours

MeV particles 10s of minutes

Recall:

$m_p = 938 \text{ MeV}/c^2$

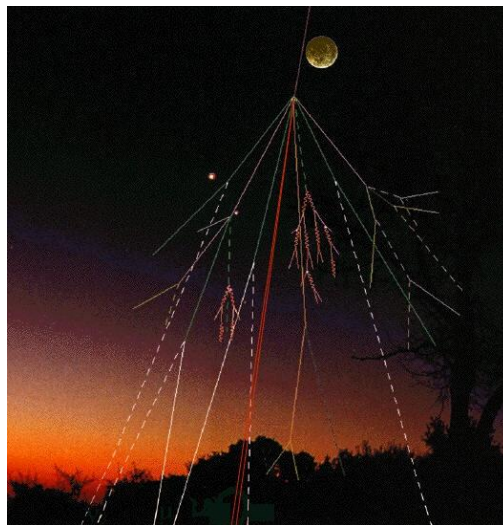
$m_e = 511 \text{ keV}/c^2$

ions are nonrelativistic,

the most energetic

electrons are relativistic

High energy cosmic rays penetrate through the magnetosphere



## Penetration of cosmic rays to the Earth's atmosphere

Recall the rigidity (momentum per charge)  $R = |p/q|$   
 In cosmic ray physics units with  $c = 1$  are often in use, thus  $[R] = \text{V}$

Total (relativistic) energy:  $W_T^2 = p^2 c^2 + m_0^2 c^4$

A little calculation gives  $R = \frac{A}{Z} [(\gamma^2 - 1)^{1/2}] W_{0A}$

atomic number
rest mass energy per nucleon

charge

If the rigidity is known, the velocity can be solved from  $\gamma = \left[ \left( \frac{RZ}{A/W_{0A}} \right)^2 + 1 \right]^{1/2}$

A cosmic **ray cut-off rigidity** is defined as the minimum rigidity for a particle to be observable at a given position in the geomagnetic field arriving from a given direction. Note that due to large energies and thus large gyroradii, the GS approximation is of no use

Størmer's formula:

$$R_c = M \frac{\cos^4 \lambda}{r^2 [1 + (1 - \sin \epsilon \sin \phi \cos^3 \lambda)^{1/2}]^2}$$

dipole moment
geomagnetic latitude

distance from the dipole
zenith angle
azimuthal angle measured from the direction of geomagnetic north wrt. geographic north

Inserting the Earth's dipole moment and expressing  $r$  in  $R_E$ , the cut-off rigidity for vertical incidence reduces to

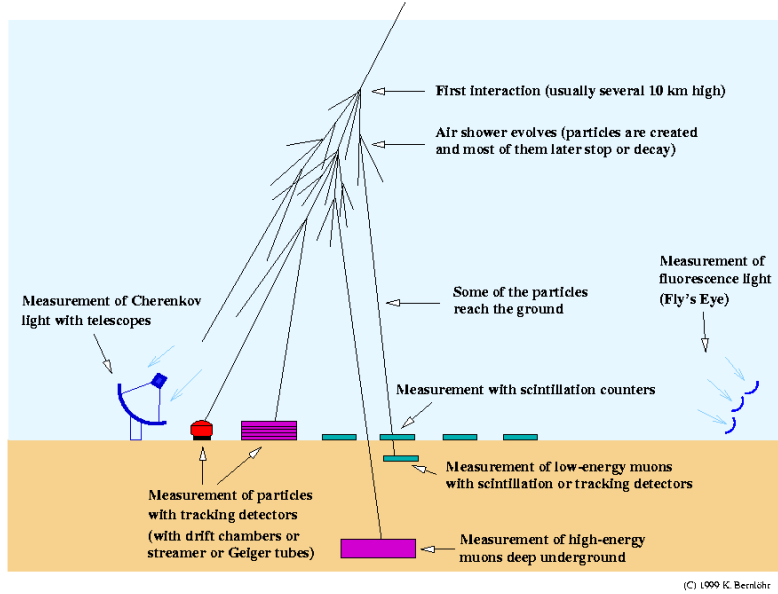
$$R_c(\text{GV}) = \frac{14.9 \cos^4 \lambda}{r^2}$$

$R_c$  at sea level at equator: 13 – 17 GV
In Helsinki: 1.4 GV
(here deviations from dipole are included)

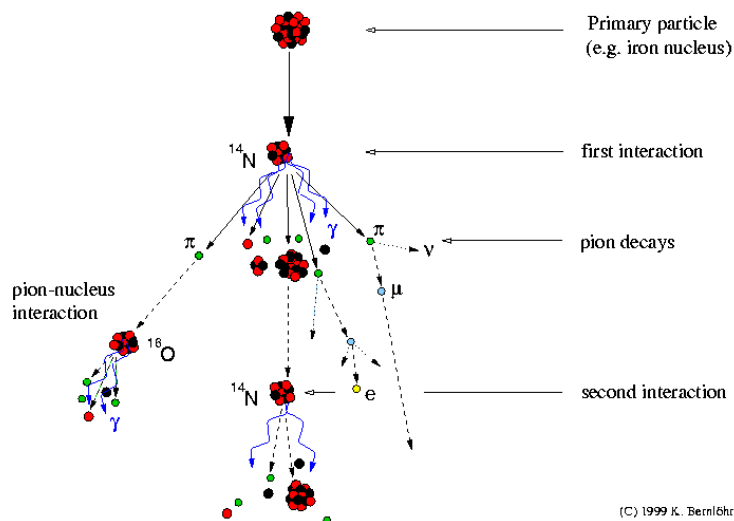
For back of the envelope estimates in the inner magnetosphere ( $L \approx 4$ ):

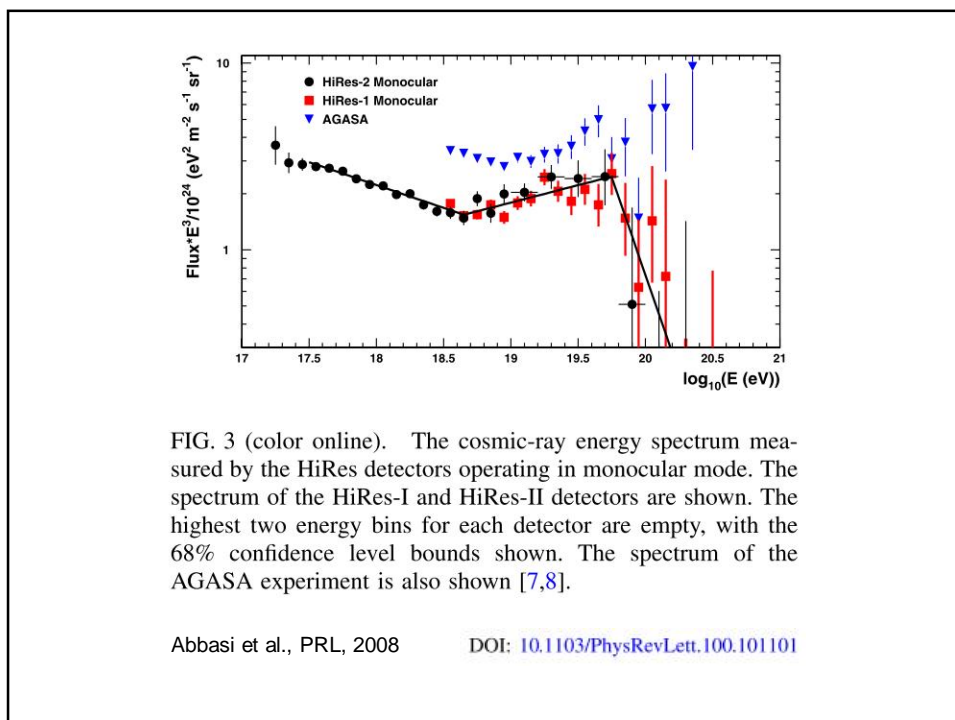
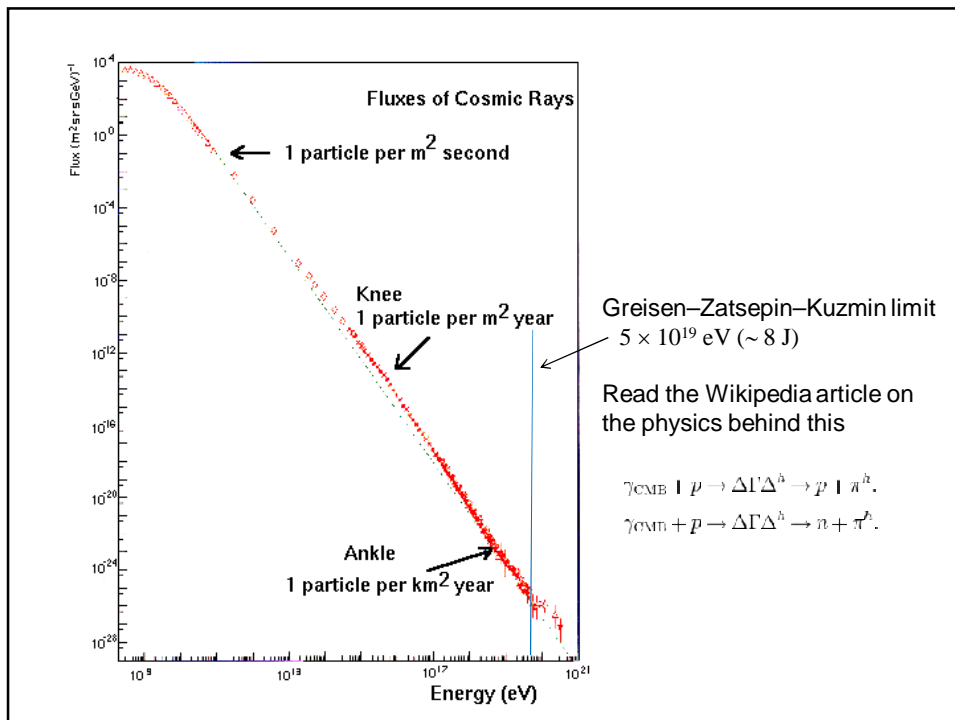
$$R_c \approx 16 L^{-2}$$

### Measuring cosmic-ray and gamma-ray air showers



### Development of cosmic-ray air showers





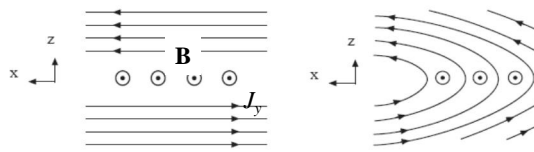
## Motion near a current sheet

Current sheets are important in plasma physics. They separate different plasma domains and they are sites of the most important energy release process, magnetic reconnection

The **Harris model** is a 2D example:

$$\mathbf{B} = B_0 \tanh\left(\frac{z}{L}\right) \mathbf{e}_x + B_n \mathbf{e}_z \quad ; \quad B_n \text{ and } B_0 \text{ are constant and } B_n \ll B_0$$

The current is according to Ampère's law  $J_y = \left(\frac{B_0}{\mu_0 L}\right) \operatorname{sech}^2\left(\frac{z}{L}\right)$



One-dimensional and two-dimensional Harris models

In one dimensional case the Harris sheet is in magnetohydrostatic equilibrium (discussed later)  $\mathbf{J} \times \mathbf{B} = \nabla p$

$$\Rightarrow p(z) = p_0 \operatorname{sech}^2\left(\frac{z}{L}\right)$$

The magnetic field is given by the vector potential

$$A_y(x, z) = -B_0 F(z) + B_n x \quad \text{Exercise: Find } F(z)$$

Consequently the particles move in the potential

$$U(x, z) = \frac{1}{2m} [P_y - qA_y(x, z)]^2$$

2-D Harris sheet is often approximated as  $\mathbf{B} = B_0 \left(\frac{z}{L}\right) \mathbf{e}_x + B_n \mathbf{e}_z$

where the field lines are parabolae  $x = \frac{B_0}{2B_n L} z^2 + \text{constant}$

In 1-D case a useful simplifying approximation is

$$\begin{aligned} B_x &= B_0 & ; & \quad z \geq L \\ B_x &= \frac{B_0 z}{L} & ; & \quad L \geq z \geq -L \\ B_x &= -B_0 & ; & \quad z \leq -L \end{aligned}$$

In the field

$$\begin{cases} B_x = B_0 & ; z \geq L \\ B_x = \frac{B_0 z}{L} & ; L \geq z \geq -L \\ B_x = -B_0 & ; z \leq -L. \end{cases}$$

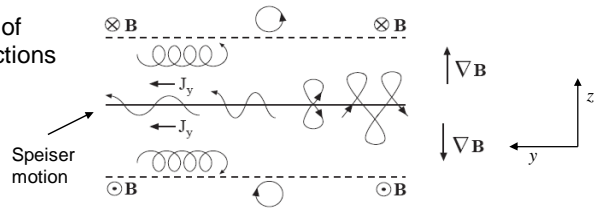
the components of the equation of motion are (exercise):

$$\begin{cases} \ddot{x} = 0 \\ \ddot{y} = \left(\frac{qB_0}{mL}\right) z \dot{z} \\ \ddot{z} = -\left(\frac{qB_0}{mL}\right) z \dot{y} \end{cases} \Rightarrow \frac{d}{dt}(y^2 + z^2) = 0 \quad (\text{conservation of energy!})$$

Motion in the z direction:  $\dot{z}^2 = (1 - k^2 + k^2 z^2)(1 - z^2)$

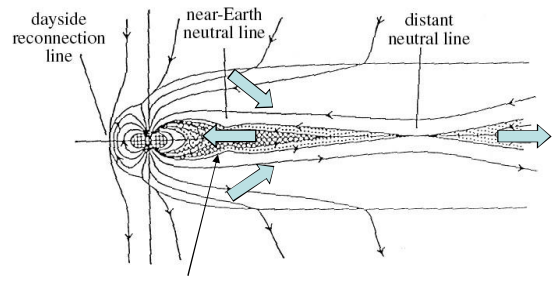
$k^2 = qB_0 z_m^2 / 4Ly_0$   
 point on the z-axis where the particle turns back      initial value of y

This gives an elliptic integral  
 → the result can be expressed in terms of Jacobi's elliptic functions

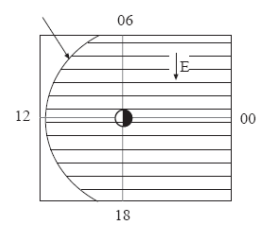


## Neutral sheet with a constant electric field

The real current sheets are not in a hydrostatic equilibrium.  
 E.g. the plasma in the Earth's magnetotail is in "convective motion."



dawn-to-dusk pointing  $E = E_0 e_y$   
 $\Rightarrow E \cdot J > 0$   
 → acceleration at the expense of the electromagnetic field



Now the components of the equation of motion are (exercise):

$$\begin{array}{l|l} \ddot{x} = 0 & c_1 = qB_0/mL \\ \ddot{y} = c_1 z \dot{z} + c_2 & c_2 = qE_0/m \\ \ddot{z} = -c_1 z \dot{y}, & \end{array}$$

and the energy equation

$$\frac{1}{2}(\dot{y}^2 + \dot{z}^2) - c_2 y = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) - c_2 y_0 \quad (\text{RHS gives the initial values})$$

These give the motion in the  $z$ -direction as:

$$\ddot{z} = -c_1 z \left[ \dot{y}_0 + \left( \frac{c_1}{2} \right) (z^2 - z_0^2) + c_2 t \right] \quad \text{An equation with chaotic solutions}$$

Assume that the particle remains in the current sheet much longer than its gyro period. Then we can rewrite the equation for large  $t$

$$\ddot{z} \approx -c_1 c_2 z t = - \left( \frac{q}{m} \right)^2 \left( \frac{E_0 B_0}{L} \right) z t$$

This type of differential equations have solutions in terms of Bessel functions.

The Bessel functions can be approximated for large  $t$  using trigonometric functions

$$z \approx - \frac{t^{-1/4}}{(E_0 B_0 / L)^{1/12} (q/m)^{1/6}} \times \left\{ A \cos \left[ \frac{2}{3} \left( \frac{q}{m} \right) \left( \frac{B_0 E_0}{L} \right)^{1/2} t^{3/2} \right] + B \sin \left[ \frac{2}{3} \left( \frac{q}{m} \right) \left( \frac{B_0 E_0}{L} \right)^{1/2} t^{3/2} \right] \right\}$$

For large  $t$  the oscillations are damped as  $t^{-1/4}$

Next we can integrate the motion in the  $y$ -direction

$$y \approx y_0 + \left[ \dot{y}_0 - \left( \frac{c_1}{2} \right) z_0^2 \right] t + \frac{c_2 t^2}{2} \quad \Rightarrow \text{the energy increases as } t^2$$

$$\frac{1}{2}(\dot{y}^2 + \dot{z}^2) - c_2 y = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) - c_2 y_0$$

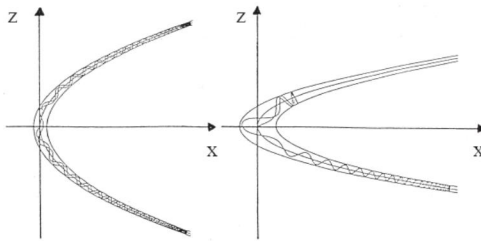
The motion is damped oscillation in  $z$ -direction (across the current sheet)  
And accelerated motion in the  $y$ -direction (ions to  $+y$  and electrons to  $-y$ )

## Harris sheet with a small $B_n$

$$\mathbf{B} = B_0 \left( \frac{z}{L} \right) \mathbf{e}_x + B_n \mathbf{e}_z \quad \& \quad \text{electric field in the } y\text{-direction} \quad \mathbf{E} = -\nabla\varphi$$

$$\begin{array}{l} \ddot{x} = c_3 \dot{y} \\ \ddot{y} = -c_3 \dot{x} + c_1 z \dot{z} + c_2 \\ \ddot{z} = -c_1 z \dot{y}, \end{array} \quad \left| \quad \begin{array}{l} c_1 = qB_0/mL \\ c_2 = qE_0/m \\ c_3 = qB_0 B_n/m. \end{array} \right.$$

$$\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + q\varphi = \frac{m}{2}(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) + q\varphi_0$$



When a dipole field is stretched, a current-sheet is introduced. For particles, whose  $R_L \rightarrow R_C$  across the current sheet, regular motion becomes non-adiabatic, i.e.,  $\mu$  is no more conserved. An example of transition to chaos

## Motion in time-dependent electric fields

Charged particles in time-dependent fields  $\rightarrow$  wave-particle interactions

Recall the case of slowly varying electric field (GS approximation)

$$\partial/\partial t \ll \omega_c$$

$\rightarrow$  polarization drift and current

$$\mathbf{v}_P = -\frac{m}{qB^2} \frac{d\mathbf{v}_E}{dt} \times \mathbf{B} = \frac{1}{\omega_c B} \frac{d\mathbf{E}_\perp}{dt} \quad \mathbf{J}_P = nq\mathbf{v}_P$$

Let  $\mathbf{E}$  increase, then  $\mathbf{J}_P \cdot \mathbf{E} > 0$  i.e., particles gain energy

This energy gain is equal to the change in the  $\mathbf{E} \times \mathbf{B}$  drift velocity (as it should)

$\mathbf{v}_E$  is a zero order drift, whereas  $\mathbf{v}_P$  is of the first order:

$$\frac{dE}{dt} \sim \omega E \Rightarrow \frac{v_P}{v_E} \sim \frac{1}{\omega_c B} \omega E \frac{B}{E} \sim \frac{\omega}{\omega_c} \ll 1$$

## Time-variations in resonance with gyro motion

Resonances can be very powerful, recall the Tacoma Bridge collapse in 1940  
Google: Tacoma Bridge → URL in Youtube:

<http://www.youtube.com/watch?v=3mclp9QmCGs>

Assume  $E \propto \exp(-i\omega t)$  &  $\omega \approx \omega_c$  & static homogeneous  $\mathbf{B}$

$$\Rightarrow \frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{E} e^{-i\omega t} + \mathbf{v} \times \mathbf{B})$$

Look for solution of the form  $\mathbf{v} = \mathbf{v}_e \exp(-i\omega t) + \mathbf{v}_m$

$$\frac{d\mathbf{v}_m}{dt} - i\omega \mathbf{v}_e e^{-i\omega t} = \frac{q}{m}(\mathbf{E} e^{-i\omega t} + \mathbf{v}_m \times \mathbf{B} + \mathbf{v}_e \times \mathbf{B} e^{-i\omega t})$$

The "magnetic" part of the eq of motion

$$\frac{d\mathbf{v}_m}{dt} = \frac{q}{m}(\mathbf{v}_m \times \mathbf{B}) \rightarrow \text{Larmor motion}$$

The electric part is

$$\frac{q}{m}\mathbf{E} = (-i\omega + \frac{q}{m}\mathbf{B} \times)\mathbf{v}_e \equiv -(i\omega + \omega_c \times)\mathbf{v}_e$$

Note that we treat the gyro frequency as a vector

$$\omega_c = -q\mathbf{B}/m$$

Multiply this from the left by  $(i\omega - \omega_c \times)$

$$\text{p} \quad \frac{q}{m}(i\omega - \omega_c \times)\mathbf{E} = (\omega^2 - \omega_c^2)\mathbf{v}_e + (\omega_c \cdot \mathbf{v}_e)\omega_c$$

Decompose this into parallel and perpendicular components

$$\mathbf{v}_{e\parallel} = \frac{i q \mathbf{E}_{\parallel}}{\omega m}$$

$$\mathbf{v}_{e\perp} = \frac{q}{m} \left( \frac{i\omega - \omega_c \times}{\omega^2 - \omega_c^2} \right) \mathbf{E}_{\perp}$$

$\mathbf{v}_{e\parallel}$  oscillates with 90° phase shift after  $\mathbf{E}_{\parallel}$

The perpendicular velocity is decomposed to two components

$$\mathbf{v}_{eL} = \frac{q}{m} \frac{i}{\omega - \omega_c} \mathbf{E}_L \quad \text{where} \quad \mathbf{E}_L = -\frac{1}{2} \left( \mathbf{E}_{\perp} + i \frac{\omega_c \times \mathbf{E}_{\perp}}{\omega_c} \right)$$

$$\mathbf{v}_{eR} = \frac{q}{m} \frac{i}{\omega + \omega_c} \mathbf{E}_R \quad \mathbf{E}_R = -\frac{1}{2} \left( \mathbf{E}_{\perp} - i \frac{\omega_c \times \mathbf{E}_{\perp}}{\omega_c} \right)$$

left-hand and right-hand polarized components

$\omega \rightarrow \omega_c$  ( $q > 0$ ) resonance btw. positive charge LH-polarization

$\omega \rightarrow \omega_c$  ( $q < 0$ ) resonance btw. negative charge and RH-polarization

## High-frequency fields "Oscillation center approximation"

Now "high-frequency" means  $\omega \gg \omega_c$

In **zero order** assume  $\mathbf{E}$  spatially homogeneous and  $\propto \exp(-i\omega t)$

The equation of motion is familiar from elementary mechanics:

$$(*) \quad \frac{d^2 \mathbf{r}}{dt^2} = \frac{q}{m} (\mathbf{E} e^{-i\omega t}) \quad \text{with solutions} \quad \mathbf{r} = \frac{q}{m\omega^2} (\mathbf{E} e^{-i\omega t}) + \mathbf{c}_1 t + \mathbf{c}_2$$

In **the first order** problem we include the magnetic field and let both  $\mathbf{E}$  and  $\mathbf{B}$  be weakly inhomogeneous and have the same time dependence

$$(**) \quad \frac{d^2 \mathbf{r}}{dt^2} = \frac{q}{m} \left[ \mathbf{E}(\mathbf{r}) + \frac{d\mathbf{r}}{dt} \times \mathbf{B}(\mathbf{r}) \right] e^{-i\omega t}$$

In the oscillation center approximation we consider (\*) as the zero-order approximation of (\*\*). This requires the following:

Condition 1:

In the Taylor series  $\mathbf{E}(\mathbf{r}) = \mathbf{E}(\mathbf{r}_0) + (\mathbf{r}_1 \cdot \nabla_0) \mathbf{E} + \dots$

$\mathbf{E}(\mathbf{r}_0)$  dominates

$\mathbf{r}_0$  is the center of oscillation and  $\mathbf{r}_1 = \mathbf{r} - \mathbf{r}_0$  oscillates

$\mathbf{r}_0$  moves slowly and  $\mathbf{E}(\mathbf{r}_0)$

is almost constant during one oscillation  $\frac{|(\dot{\mathbf{r}}_0 \cdot \nabla) \mathbf{E}|}{\omega} \ll E$

Condition 2:

$(d\mathbf{r}/dt) \times \mathbf{B}$  is small, i.e.,  $\omega \gg \omega_c$

$d^2 \mathbf{r} / dt^2 \sim \omega d\mathbf{r}_1 / dt$   $\Rightarrow$  the magnetic term is proportional to  $\omega_c d\mathbf{r} / dt$

The speed  $d\mathbf{r} / dt$  must not be much larger than  $d\mathbf{r}_1 / dt$

Under these conditions we can expand

$$\begin{aligned} \frac{d^2 \mathbf{r}_0}{dt^2} + \frac{d^2 \mathbf{r}_1}{dt^2} &= \frac{q}{m} \left[ \mathbf{E}(\mathbf{r}_0) + (\mathbf{r}_1 \cdot \nabla_0) \mathbf{E} + \frac{d\mathbf{r}_0}{dt} \times \mathbf{B}(\mathbf{r}_0) + \frac{d\mathbf{r}_1}{dt} \times \mathbf{B}(\mathbf{r}_0) \right] \\ &+ \frac{q}{m} \left[ \frac{d\mathbf{r}_0}{dt} \times (\mathbf{r}_1 \cdot \nabla_0) \mathbf{B} + \frac{d\mathbf{r}_1}{dt} \times (\mathbf{r}_1 \cdot \nabla_0) \mathbf{B} \right] \quad \leftarrow \text{This line is of the second order} \end{aligned}$$

$$\frac{d^2 \mathbf{r}_0}{dt^2} + \frac{d^2 \mathbf{r}_1}{dt^2} = \frac{q}{m} \left[ \mathbf{E}(\mathbf{r}_0) + (\mathbf{r}_1 \cdot \nabla_0) \mathbf{E} + \frac{d\mathbf{r}_0}{dt} \times \mathbf{B}(\mathbf{r}_0) + \frac{d\mathbf{r}_1}{dt} \times \mathbf{B}(\mathbf{r}_0) \right] + \frac{q}{m} \left[ \frac{d\mathbf{r}_0}{dt} \times (\mathbf{r}_1 \cdot \nabla_0) \mathbf{B} + \frac{d\mathbf{r}_1}{dt} \times (\mathbf{r}_1 \cdot \nabla_0) \mathbf{B} \right]$$

These are the largest order (zero-order) terms

Zero order:  $\mathbf{r}_1 = -\frac{q}{m\omega^2} \mathbf{E}_0$

First order: We consider averages of the oscillation period

$\langle d\mathbf{r}_0/dt \times \mathbf{B}_0 \rangle$  can be neglected, because  $d\mathbf{r}_0/dt$  is small and  $\langle \mathbf{B}_0 \rangle = \bar{0}$

$$\Rightarrow \left\langle \frac{d^2 \mathbf{r}_0}{dt^2} \right\rangle = \frac{q}{m} \left\{ \langle (\mathbf{r}_1 \cdot \nabla_0) \mathbf{E} \rangle + \left\langle \frac{d\mathbf{r}_1}{dt} \times \mathbf{B}_0 \right\rangle \right\} \approx d^2 \mathbf{r}_0 / dt^2$$

Finally insert the zero order solution  $\mathbf{r}_1$  and we find the motion of the oscillation center:

$$\frac{d^2 \mathbf{r}_0}{dt^2} = -\frac{q^2}{m^2 \omega^2} \left\langle \nabla_0 \frac{E^2}{2} \right\rangle - \frac{q^2}{m^2 \omega^2} \left\langle \frac{\partial}{\partial t} (\mathbf{E}_0 \times \mathbf{B}_0) \right\rangle$$

The oscillation center is accelerated  $= 0$  for a standing wave by the "ponderomotive" force  $\propto -\nabla \Phi$

where the ponderomotive potential is  $\Phi = \frac{q^2}{m^2 \omega^2} \left\langle \frac{E^2}{2} \right\rangle$

The ponderomotive force is conservative

$$\frac{1}{2} \left( \frac{dr_0}{dt} \right)^2 + \Phi = \text{constant} \Rightarrow \frac{1}{2} \left[ \left( \frac{dr_0}{dt} \right)^2 + \left\langle \left( \frac{dr_1}{dt} \right)^2 \right\rangle \right] = \text{constant}$$

Energy oscillates between translation and oscillation

Application: A packet of EM waves propagates in a plasma at a frequency that exceeds the plasma frequency. If the wave packet is sufficiently intense, plasma particles are repelled from the centre of the packet by the ponderomotive force. The resulting variation in the plasma density gives rise to a cylindrical well in the index of refraction which acts as a wave-guide for the EM wave.