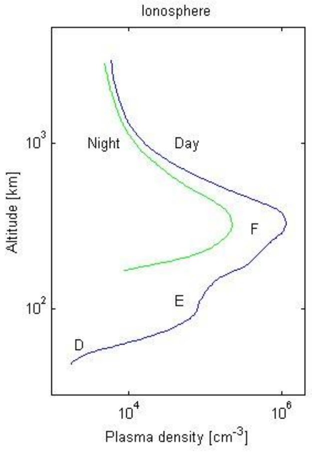
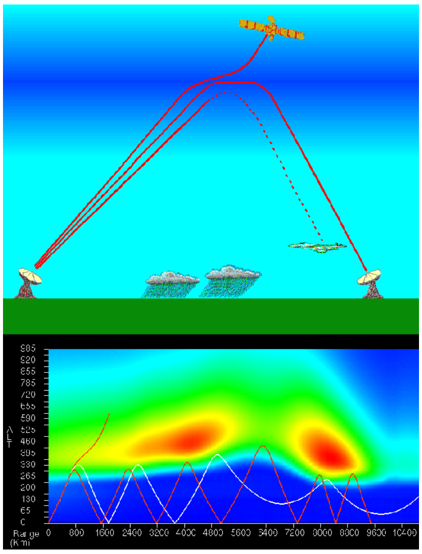
Wave propagation

- Waves are important in space research
  - remote observations from radio waves to gamma rays
    - the signals propagate through space plasma
  - wave diagnostics of medium
    - ionospheric radars, Faraday rotation, etc.
- Waves in space plasmas
  - wide variety of wave modes
  - natural response of plasma to perturbations
  - specific frequencies
    - plasma frequency, gyro frequency and their hybrids
  - energy transfer
    - particle acceleration / heating
    - instabilities

Motivation for us





EM waves in vacuum

In absence of charges and currents $(\rho = 0;\ \mathbf{J} = 0)$
Maxwell's equations reduce to

$$\begin{aligned} \nabla \times \mathbf{E} &= -\mu_0\, \partial \mathbf{H}/\partial t \\ \nabla \times \mathbf{H} &= +\epsilon_0\, \partial \mathbf{E}/\partial t \end{aligned} \qquad \mathbf{H} = \mathbf{B}/\mu_0$$

$$\Rightarrow \begin{cases} \nabla^2 \mathbf{H} - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \\ \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \end{cases}$$

homogeneous wave equations

propagation speed: $c = \dfrac{1}{\sqrt{\epsilon_0 \mu_0}}$

Consider propagation to ±z – direction and plane wave solutions.
For a plane wave there is a plane propagating with the wave where the electric field is constant. Such a solution can be written as

$E_x(z,t) = E_0 \cos(kz - \omega t)$ or in vector form $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$

E_0 amplitude
$\omega = 2\pi f$ angular frequency
$k = 2\pi/\lambda$ wave number

$c = \dfrac{\omega}{k}$ phase speed

A wave can be considered as a plane wave only far from the source.
Sometimes it is necessary to consider spherical waves, i.e., waves, for which there exists a spherical surface where $\mathbf{E}$ is constant.

An example is the field of a radiating dipole:

$$\mathbf{E}(r,\theta,\phi,t) = \frac{a}{r}\sin\theta\cos(kr - \omega t)\mathbf{e}_\theta$$

(this is an approximation up to terms of the order of $1/r^3$; the exact solution can be expressed in terms of Bessel functions)

Plane waves are most convenient to treat using the complex notation

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ \mathbf{B} &= \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \end{aligned}$$

Now the differential operations reduce to multiplications

$$\begin{aligned} \nabla \cdot \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} &= i\mathbf{k} \cdot \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ \nabla \times \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} &= i\mathbf{k} \times \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ \frac{\partial}{\partial t}\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} &= -i\omega \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \end{aligned}$$

and Maxwell's equations become algebraic equations

$$\begin{aligned} i\mathbf{k}\cdot\mathbf{D} &= \rho \\ \mathbf{k}\cdot\mathbf{B} &= 0 \\ \mathbf{k}\times\mathbf{E} &= \omega \mathbf{B} \\ i\mathbf{k}\times\mathbf{H} &= \mathbf{J} - i\omega\mathbf{D} \end{aligned}$$

From vacuum to dielectric media

If there are no free charges or currents, but $\varepsilon \neq \varepsilon_0$ and $\mu \neq \mu_0$ (but constants)

Now $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \rightarrow v = \frac{1}{\sqrt{\epsilon \mu}}$

$n = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}}$ is the index of refraction

$k = \frac{\omega}{v} = \sqrt{\epsilon \mu}\, \omega = \frac{n}{c}\omega$ dispersion equation

$v_p = \omega / k = c/n$ phase velocity (propagation of the constant phase)

$v_g = \partial\omega / \partial k = c/n$ group velocity (propagation of energy and information!)

If in addition $\mathbf{J} = \sigma\mathbf{E}$ (σ constant), the situation is more complicated

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu\sigma\mathbf{E} + \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}$$

$$\Rightarrow \quad \nabla^2\mathbf{E} - \mu\sigma\frac{\partial \mathbf{E}}{\partial t} - \mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

one form of so-called telegrapher's equations;
a standard class-room example of using Fourier transformations to solve partial differential equations

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Instead of making the Fourier transformations, let's start from Maxwell's equations and make the plane wave assumption

$$\mathbf{k}\cdot\mathbf{E} = 0$$
$$\mathbf{k}\cdot\mathbf{H} = 0$$
$$\mathbf{k}\times\mathbf{E} = \omega\mu\mathbf{H}$$
$$i\mathbf{k}\times\mathbf{H} = (\sigma - i\omega\epsilon)\mathbf{E}$$

Clearly $\mathbf{k}$, $\mathbf{E}$ and $\mathbf{H}$ are all perpendicular to each other: transversal wave

Choose: $\mathbf{k} \parallel \mathbf{e}_z$, $\mathbf{E} \parallel \mathbf{e}_x$, $\mathbf{H} \parallel \mathbf{e}_y$

$$\Rightarrow \quad \begin{aligned} kE_x &= \omega\mu H_y \\ ikH_y &= -(\sigma - i\omega\epsilon)E_x \end{aligned} \quad \Rightarrow \quad k^2 = \epsilon\mu\omega^2 + i\sigma\mu\omega$$

dispersion equation

now $v_p \neq v_g$!

write $k = |k|\exp(i\alpha) \Rightarrow \begin{cases} |k| = \sqrt{\mu\omega\sqrt{\epsilon^2\omega^2 + \sigma^2}} \\ \alpha = \frac{1}{2}\arctan(\frac{\sigma}{\epsilon\omega}) . \end{cases}$

The solution is $\mathbf{E} = E_0\mathbf{e}_x \exp[i(|k|(\cos\alpha)z - \omega t)]\exp[-|k|(\sin\alpha)z]$

$$v_p = \frac{\omega}{\mathrm{Re}(k)} = \frac{\omega}{|k|\cos\alpha}$$

select the phase of α so that the wave is damped!

$$\delta = \frac{1}{\mathrm{Im}(k)} = \frac{1}{|k|\sin\alpha} .$$

skin depth

impedance of the medium:

$$Z = \frac{E_x}{H_y} = \frac{\mu\omega}{k} = \sqrt{\frac{\mu\omega}{\sqrt{\epsilon^2\omega^2 + \sigma^2}}}\exp[-\frac{i}{2}\arctan(\frac{\sigma}{\epsilon\omega})]$$

Examples

Good conductor: $\sigma >> \epsilon\omega \ \Rightarrow \alpha = 45^\circ;\ \delta = \sqrt{\frac{2}{\mu\sigma\omega}}\,.$

$v_p = \delta\omega\tan\alpha = \delta\omega$

$$\text{Cu:}\quad \begin{array}{lll} f = 50\ \text{Hz} & \delta = 1\ \text{cm} & v_p = 3\ \text{m/s} \\ f = 50\ \text{MHz} & \delta = 10\ \mu\text{m} & v_p = 3\cdot 10^3\ \text{m/s} \end{array}$$

$Z = \sqrt{\frac{\mu\omega}{\sigma}}e^{-i\pi/4} \ \Rightarrow 45^\circ$ phase shift between $\mathbf{E}$ and $\mathbf{H}$.

Non-conducting medium: $\sigma = 0,\ \epsilon > 0,\ \mu = \mu_0$

$\Rightarrow \alpha = 0$, i.e., the wave is not damped.

$$Z = \sqrt{\frac{\mu_0}{\epsilon}} \equiv Z_0\sqrt{\frac{\epsilon_0}{\epsilon}}\,,$$

Z_0 is called **vacuum impedance**: $\sqrt{\frac{\mu_0}{\epsilon_0}} = 376.73\ \Omega$

Wave polarization

Consider an EM wave propagating in $+z$ direction
Focus on the plane $z = 0$, denote $\rho = E_y/E_x = -H_x/H_y$; $\rho = |\rho|\exp(i\alpha)$

1) If ρ is real, E_y and E_x are in the same phase
   Direction of $\mathbf{E}$ is $(1,\rho,0)$ (if $\rho = \infty$, $\mathbf{E}$ is in the y-direction)
   This is a linearly polarized wave
2) If $\rho = +i$, there is a phase shift ($\alpha = \pi/2$) between E_y and E_x
   This is a right-hand circularly polarized wave (positive helicity)
   $\mathbf{E} = E_0(\mathbf{e}_x + i\mathbf{e}_y)e^{i(kz-\omega t)}$
3) If $\rho = -i$, there is a phase shift ($\alpha = -\pi/2$) between E_y and E_x
   This is a left-hand circularly polarized wave (negative helicity)
   $\mathbf{E} = E_0(\mathbf{e}_x - i\mathbf{e}_y)e^{i(kz-\omega t)}$
4) If ρ is a general complex number, the wave is elliptically polarized

All polarizations can be obtained from left- and right-hand polarized waves by superposition.

WARNING: In optics the left- and right-hand polarizations are defined in the opposite way!!

$\mathbf{k}$ is a vector

Wave vector: $\mathbf{k} = k\mathbf{n}$ ← unit vector defining the direction of the wave normal $\perp$ surface of constant wave phase; it is the direction of wave propagation

In isotropic media $\mathbf{k} \parallel \mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^*$

If the medium is anistropic, it is possible that $\mathbf{S} \nparallel \mathbf{k}$

i.e., the energy does not need to propagate in the direction of wave normal.

Energy and infromation propagate with the group velocity $\mathbf{v}_g = \dfrac{\partial\omega}{\partial\mathbf{k}}$

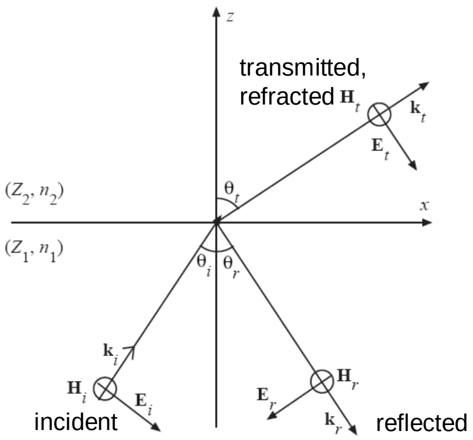
Let θ be the angle between $\mathbf{B}$ and $\mathbf{k}$ and the frequency a function of $\mathbf{k}$ and θ

$$\mathbf{v}_g = \frac{\partial\omega}{\partial\mathbf{k}} = \left.\frac{\partial\omega}{\partial k}\right|_\theta \mathbf{e}_k + \left.\frac{1}{k}\frac{\partial\omega}{\partial\theta}\right|_k \mathbf{e}_\theta$$

$$\tan\alpha = -\left.\frac{1}{k}\frac{\partial k}{\partial\theta}\right|_\omega$$

angle between $\mathbf{v}_g$ and $\mathbf{v}_p$

Reflection and refraction



$$\begin{aligned}\mathbf{k}_i &= k_i(\sin\theta_i, 0, \cos\theta_i)\\ \mathbf{k}_r &= k_r(\sin\theta_r, 0, -\cos\theta_r)\\ \mathbf{k}_t &= k_t(\sin\theta_t, 0, \cos\theta_t)\,.\end{aligned}$$

This is vertical polarization ($\mathbf{E}$ has a vertical component)
The opposite case is horizontal polarization
All polarizations can be expressed as superposition of these.

$$\begin{aligned}
\mathbf{E}_i &= E_i(\cos\theta_i, 0, -\sin\theta_i)\exp[i(k_i(\sin\theta_i x + \cos\theta_i z) - \omega t)] \\
\mathbf{H}_i &= \frac{E_i}{Z_1}(0,1,0)\exp[i(k_i(\sin\theta_i x + \cos\theta_i z) - \omega t)] \\
\mathbf{E}_r &= E_r(-\cos\theta_r, 0, -\sin\theta_r)\exp[i(k_r(\sin\theta_r x - \cos\theta_r z) - \omega t)] \\
\mathbf{H}_r &= \frac{E_r}{Z_1}(0,1,0)\exp[i(k_r(\sin\theta_r x - \cos\theta_r z) - \omega t)] \\
\mathbf{E}_t &= E_t(\cos\theta_t, 0, -\sin\theta_t)\exp[i(k_t(\sin\theta_t x + \cos\theta_t z) - \omega t)] \\
\mathbf{H}_t &= \frac{E_t}{Z_2}(0,1,0)\exp[i(k_t(\sin\theta_t x + \cos\theta_t z) - \omega t)] .
\end{aligned}$$

Boundary conditions:

$$\begin{aligned}
\mathbf{n}_{12} \times (\mathbf{E}_1 - \mathbf{E}_2) &= 0 \\
\mathbf{n}_{12} \times (\mathbf{H}_1 - \mathbf{H}_2) &= \mathbf{K}
\end{aligned}$$

Surface current induced by the wave ($\mathbf{K}$)

Assume $\mathbf{K} = 0 \Rightarrow$

$$\begin{aligned}
E_{ix} + E_{rx} &= E_{tx} \\
H_{iy} + H_{ry} &= H_{ty}
\end{aligned}$$

$$\Rightarrow \begin{cases}
E_i\cos\theta_i\exp[i(k_i\sin\theta_i x - \omega t)] - E_r\cos\theta_r\exp[i(k_r\sin\theta_r x - \omega t)] \\
= E_t\cos\theta_t\exp[i(k_t\sin\theta_t x - \omega t)] \\
\frac{E_i}{Z_1}\exp[i(k_i\sin\theta_i x - \omega t)] - \frac{E_r}{Z_1}\exp[i(k_r\sin\theta_r x - \omega t)] \\
= \frac{E_t}{Z_2}\exp[i(k_t\sin\theta_t x - \omega t)] .
\end{cases}$$

These conditions must be filled all the time at every point on the interface $\Rightarrow$

$$\omega_i = \omega_r = \omega_t = \omega$$
$$k_i\sin\theta_i = k_r\sin\theta_r = k_t\sin\theta_t$$

As the incident and reflected waves are in the same medium, we get the results familiar from the high-school physics

$$\frac{c}{\omega}k_i = \frac{c}{\omega}k_r \;\Rightarrow\; k_i = k_r \;\Rightarrow\; \theta_i = \theta_r$$

$$\sin\theta_t = \frac{k_i}{k_t}\sin\theta_i = \frac{n_1}{n_2}\sin\theta_i \qquad \text{Snell's law}$$

Reflection coefficient for vertical polarization

$$R_{\parallel} = \frac{E_r}{E_i} = \frac{Z_1 \cos\theta_i - Z_2 \cos\theta_t}{Z_1 \cos\theta_i + Z_2 \cos\theta_t}$$

In our examples $\mu_1 = \mu_2 (= \mu_0) \quad \Rightarrow \quad Z_1/Z_2 = n_2/n_1$

$$\Rightarrow \quad R_{\parallel} = \frac{E_r}{E_i} = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$

$$T_{\parallel} = \frac{E_t}{E_i} = \frac{2 n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t}.$$

Fresnel’s formulas
for vertical polarization

transmission coefficient for vertical polarization

$$R_{\perp} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

$$T_{\perp} = \frac{2 n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}.$$

Fresnel’s formulas
for horizontal polarization

Example: Total reflection from the ionosphere

For radio waves of sufficiently high frequency $(\omega \gg \omega_{ce}\ ,\ \omega \gg \nu_{coll}\ ,\ \omega \gg \omega_p)$

The air below the ionosphere is a good non-conductive dielectric

$$\sigma = 0,\ \mu = \mu_0,\ n_1 = 1$$

In the ionosphere $\quad n_2 = \frac{ck}{\omega} = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} < 1$

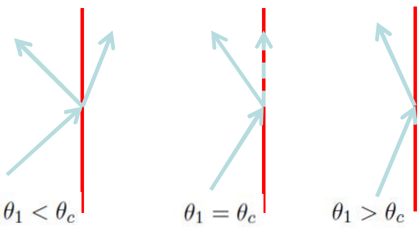
Fresnel’s formulas $\Rightarrow \quad |R_{\perp}| \geq |R_{\parallel}|$ for all angles of incidence

Thus the horizontal polarization is more efficient in radio communications via the ionosphere than the vertical polarization

For sufficiently large θ_i

$$\sin\theta_t = \frac{n_1}{n_2} \sin\theta_i \geq 1 \quad \Rightarrow \quad |R_{\perp}| = |R_{\parallel}| = 1$$

i.e., the whole wave is reflected;
total reflection

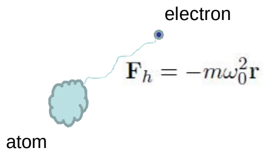


Dispersion of the medium
(so-called Drude-Lorentz model)

In dispersive media $\epsilon = \epsilon(\omega, \mathbf{k})$ (in plasmas we can usually assume $\mu = \mu_0$)

Consider the motion of an electron bound to an atom by a harmonic force

$$\mathbf{F}_h = -m\omega_0^2\mathbf{r}$$



Include a frictional force to damp the harmonic oscillations $\mathbf{F}_d = -m\gamma\frac{d\mathbf{r}}{dt}$

Under the influence of an external electric field the eq. of motion becomes

$$m\left(\frac{d^2\mathbf{r}}{dt^2} + \gamma\frac{d\mathbf{r}}{dt} + \omega_0^2\mathbf{r}\right) = -e\mathbf{E}(\mathbf{r}, t)$$

Let the time dependences be harmonic $\propto \exp(-i\omega t)$ $\Rightarrow$

$$\mathbf{r} = \frac{-e\mathbf{E}}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}$$

By definition the dipole moment associated with this electron is $\mathbf{p} = -e\mathbf{r}$

Assume that in a unit volume there are N molecules and Z electrons/molecule

Let f_j electrons of each molecule have an eigenfrequency ω_{0j} and a damping factor γ_j

↖ "oscillation strengths": $\sum_j f_j = Z$

Þ the density of dipole moments, i.e., electric polarization / unit volume

$$\mathbf{P} = \frac{Ne^2\mathbf{E}}{m}\sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 - i\omega\gamma_j}$$

Write the electric displacement $\mathbf{D} \equiv \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}$ $\Rightarrow$

$$\epsilon(\omega) = \epsilon_0(1 + \chi(\omega)) = \epsilon_0\left(1 + \frac{Ne^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 - i\omega\gamma_j}\right)$$

The electric permittivity is a complex function and depends on the frequency

Let the number of free electrons / molecule be f_0
Their eigenfrequencies are $\omega_{00} = 0$

$$\epsilon(\omega) = \underbrace{\epsilon_0 \left(1 + \frac{Ne^2}{m\epsilon_0} \sum_{j \neq 0} \frac{f_j}{\omega_{0j}^2 - \omega^2 - i\omega\gamma_j}\right)}_{\epsilon_b} - \frac{Ne^2}{m\omega} \frac{f_0}{\omega + i\gamma_0}$$

conductance electrons

Ampère's law: $\nabla \times \mathbf{H} = (\sigma - i\omega\epsilon_b)\mathbf{E} \equiv -i\omega\epsilon\mathbf{E}$

$$\epsilon = \epsilon_b + \frac{i\sigma}{\omega}$$

$$\sigma = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)}$$

Example: Copper in room temperature

$\sigma = 5.6 \cdot 10^7 \ (\Omega\mathrm{m})^{-1}, \ N = 8 \cdot 10^{28} \ \mathrm{m}^{-3}, \ f_0 = 1$
$\Rightarrow \ \gamma_0 = 4 \cdot 10^{13} \ \mathrm{s}^{-1}$.

$$\left.\begin{matrix} \gamma_0 \gg |\omega| \\ f_0 = 1 \end{matrix}\right\} \Rightarrow \quad \sigma = \frac{Ne^2}{m\gamma_0} .$$

Static conductivity is a good approximation for Cu, if $|\omega| \ll 4 \cdot 10^{13} \ \mathrm{s}^{-1}$

compare with FM radio $\omega \approx 6 \cdot 10^8 \ \mathrm{s}^{-1}$

ω_{0j} are resonant frequencies

Often $\gamma_j \ll \omega_{0j} \Rightarrow \epsilon(\omega)$ almost real, except when $\omega \approx \omega_{0j}$

$$\epsilon(\omega) \approx \epsilon_0 \left(1 + \frac{Ne^2}{m\epsilon_0} \sum_{j \neq 0} \frac{f_j}{\omega_{0j}^2 - \omega^2}\right)$$

$$\omega < \omega_{0j} \quad \Rightarrow \quad \epsilon > \epsilon_0$$
$$\omega > \omega_{0j} \quad \Rightarrow \quad \epsilon < \epsilon_0$$
$$\omega = \omega_{0j} \quad \Rightarrow \quad \mathrm{Re}\ \epsilon = \epsilon_0$$

Dispersion is called
normal if $d(\mathrm{Re}\ \epsilon(\omega))/d\omega > 0$
anomalous if $d(\mathrm{Re}\ \epsilon(\omega))/d\omega < 0$

only close to resonant frequencies, where $\mathrm{Im}\,\epsilon \neq 0$.

At resonance energy is transferred between the wave and particles
Polarization current is $\mathbf{J}_P = \partial\mathbf{P}/\partial t$ and the work done by $\mathbf{E}$ is

$$W = \mathbf{E} \cdot \mathbf{J}_P = \mathbf{E} \cdot \partial\mathbf{P}/\partial t$$

Averaging over one oscillation period

$$\langle W \rangle = \frac{1}{2}\mathrm{Re}(\mathbf{E}\cdot(-i\omega\mathbf{P})^*) = \frac{1}{2}\mathrm{Re}(i\omega(\epsilon^* - \epsilon_0)\mathbf{E}\cdot\mathbf{E}^*) = \frac{\omega}{2}\mathrm{Im}\ \epsilon(\omega)|\mathbf{E}|^2$$

$\mathrm{Im}\ \epsilon > 0 \ \Rightarrow$ energy transfer from field to particles

Note that this model always has $\mathrm{Im}\,\epsilon > 0 \quad \omega > 0$ and thus cannot be applied to lasers or plasma instabilities where the wave gains energy from plasma

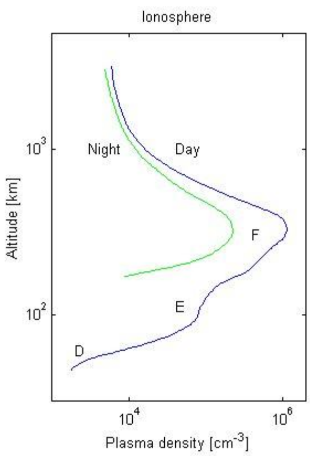
refractive index: $n(\omega) \ = \ \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \approx \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}} \ \Rightarrow$

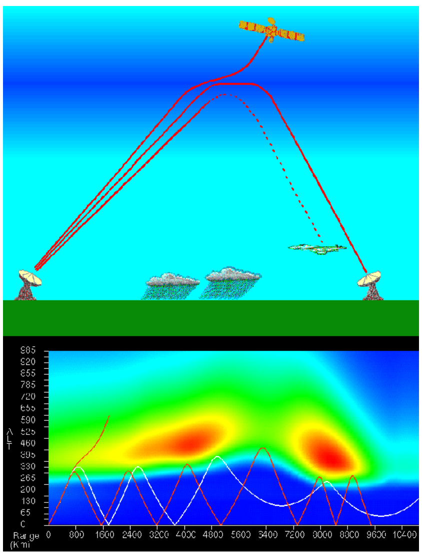
wave number: $k(\omega) \ = \ \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}}\,\frac{\omega}{c}\,.$

phase velocity: $v_p = \frac{\omega}{k} = \frac{c}{n(\omega)}$

group velocity: $v_g = \frac{d\omega}{dk} = \frac{1}{dk/dw} = \frac{c}{n(\omega) + \omega\frac{dn}{d\omega}}$

Radio wave propagation in the ionosphere





Isotropic, lossless ionosphere

Isotropic: No effects of $\mathbf{B}$ included; OK if $\omega \gg \omega_{ce} \approx 10^7\ \mathrm{s}^{-1}$

Lossless: Effects of collisions neglected; OK if $\omega \gg \nu_{coll}$

These are fulfilled if $f \gg \omega_{ce}/2\pi \approx 1.6\ \mathrm{MHz}$

Consider the electron motion in the electric field of a (plane) wave assuming that ions form an immobile background.
Now we determine ε and σ

$$m\frac{d\mathbf{v}}{dt} = -i\omega m\mathbf{v} = -e\mathbf{E} \quad \Rightarrow \quad \mathbf{J} = -n_e e\mathbf{v} = \frac{\omega_p^2}{\omega^2} i\omega\epsilon_0\mathbf{E}$$

Thus we have formally found Ohm's law with $\sigma = \frac{\omega_p^2}{\omega^2} i\omega\epsilon_0$

Assume that except for this conductivity the plasma is vacuum: $\begin{cases} \epsilon = \epsilon_0 \\ \mu = \mu_0 \end{cases}$

Now the Ampère-Maxwell law $\nabla \times \mathbf{H} = \mathbf{J} + \partial\mathbf{D}/\partial t$

yields for the plane waves

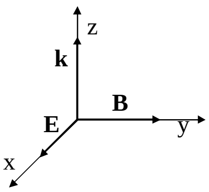
$$i\mathbf{k} \times \mathbf{H} = \frac{\omega_p^2}{\omega^2} i\omega\epsilon_0\mathbf{E} - i\omega\epsilon_0\mathbf{E} = -i\omega\left(1 - \frac{\omega_p^2}{\omega^2}\right)\epsilon_0\mathbf{E}$$

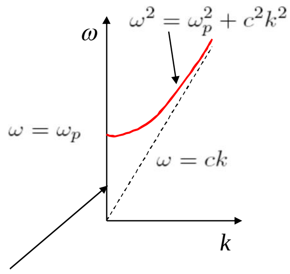
Thus the medium looks like dielectric with $\epsilon = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\epsilon_0$

In plasma physics we often write $\omega_p^2/\omega^2 \equiv X$

The index of refraction is $n = \sqrt{1-X}$ and $c = \frac{\omega}{k}\sqrt{1-X}$

$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1-X}} \qquad v_g = \frac{\partial\omega}{\partial k} = c\sqrt{1-X}$$





$$\begin{aligned} kE_x &= \omega\mu_0 H_y \\ kH_y &= \left(1 - \frac{\omega_p^2}{\omega^2}\right)\omega\epsilon_0 E_x \end{aligned}$$

$$\Rightarrow \quad \omega^2 = \omega_p^2 + c^2k^2$$

If $X > 1$, the wave does not propagate

In the ionosphere the maximum electron densities are of the order $10^{12}\ \mathrm{m}^{-3}$

$$f_{pe}(\mathrm{Hz}) \equiv \omega_{pe}/2\pi \approx 9 \cdot \sqrt{n_e(\mathrm{m}^{-3})}$$

$\Rightarrow$ the maximum plasma frequency in the ionosphere is about $9\ \mathrm{MHz}$

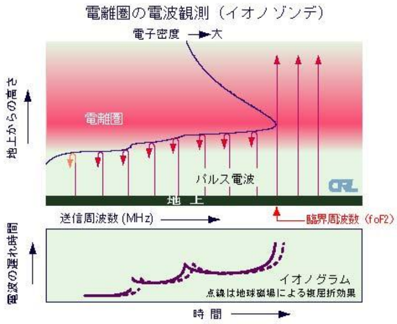
EM wave packet ($f < 9\ \mathrm{MHz}$) propagates vertically up and returns after time T

Virtual reflection height: $h' = cT/2$

In reality $T = 2\int_0^h \frac{dz}{v_g}$ ← real reflection height $\leftrightarrow$ v_g becomes zero

$$\Rightarrow \qquad h' = c\int_0^h \frac{dz}{v_g} = \int_0^h \frac{dz}{\sqrt{1 - X(z)}}$$

Ionosonde



Linear density profile

$$n_e = a(z - z_1) \quad \text{when } z > z_1$$
$$n_e = 0 \quad \text{when } z \le z_1$$

Real reflection takes place where $\omega^2 = \omega_{pe}^2$

$\Rightarrow \quad h = z_1 + \frac{\epsilon_0 m}{ae^2}\omega^2$ from which we get the virtual height

$$h' = \int_0^h \frac{dz}{\sqrt{1 - \frac{a(z - z_1)e^2}{\epsilon_0 m \omega^2}}} = z_1 + \frac{2\epsilon_0 m}{ae^2}\omega^2$$

Parabolic density profile is also straightforward to calculate (exercise) from

$$n_e = n_m\left[1 - \left(\frac{z - z_m}{a}\right)^2\right] \quad \text{when } |z - z_m| < a$$
$$n_e = 0 \quad \text{when } |z - z_m| \ge a$$

m refers to maximum density

Oblique propagation

θ_0: angle between the vertical direction and $\mathbf{k}$

The horizontal distance of wave packet propagation $y = ct\sin\theta_0$

The vertical motion is obtained by substitution $\omega \rightarrow \omega\cos\theta_0$

the real height at time t

$$h'(t) = ct\cos\theta_0 = \int_0^z \frac{dz'}{\sqrt{1 - \dfrac{\omega_{pe}^2(z')}{\omega^2\cos^2\theta_0}}}$$

Eliminating *t*: $$y = \sin\theta_0 \int_0^z \frac{dz'}{\sqrt{\cos^2\theta_0 - \dfrac{\omega_{pe}^2(z')}{\omega^2}}}$$

This gives the ”ray path”.
In isotropic plasma the ray propagates in the direction of $\mathbf{n}$

Weakly inhomogeneous ionosphere
(the WKB approximation)

What happens at the reflection point, i.e., where
the vertical component of the group velocity is zero? $n = \sin\theta_0$

Reflection is expected to take place at each interface of different *n*

Consider a wave with $\omega = 2\,\omega_{pe}$ and (for simplicity) vertical propagation $\theta_i = 0$

$n = \sqrt{1 - \omega_{pe}^2/\omega^2}$ þ $R = 0.07$ reflection should be easy to observe
but there is no reflection!!

Construct a model ionosphere consisting of
of thin layers with upward increasing density, i.e.,
decreasing index of refraction

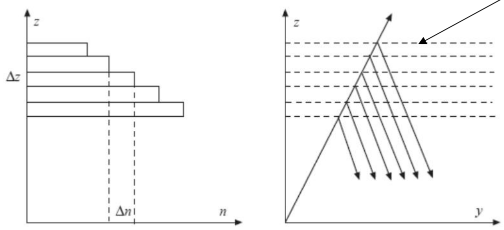


Figure is for oblique propagation but let $\theta_i = \theta_t = 0$

At each layer

$$R = \frac{n_1 - n_2}{n_1 + n_2} \approx \frac{\triangle n}{2n}$$

but this is not the whole story.
The phase of the wave at each layer becomes critical

Let E_0 be the amplitude for the incident wave in the neutral atmosphere
The electric field of the wave after reflection is $E = E(z)$
From each layer
$(\triangle n/2n)E(z)$ is reflected
$(1-(\triangle n/2n))E(z)$ is refracted
$\triangle n < 0 \Rightarrow$ the electric field of the refracted wave grows!

The wave propagates toward an increasing impedance $Z = E/H$
For the wave magnetic field

$$\frac{H_t}{H_i} = T\frac{Z_1}{Z_2} = T\frac{n_2}{n_1} \approx 1 + \frac{\triangle n}{2n} < 1$$

At the limit of continuous profile $\triangle z \to 0$

$$E + dE = \left(1 - \frac{dn}{2n}\right)E \qquad \Rightarrow \qquad E = \frac{E_0}{\sqrt{n}} \; ; \; H = \sqrt{n}H_0$$

At each layer the phase of the wave is shifted by $kdz = nk_0dz$
At the altitude z the accumulated retardation is given by

$$\int_0^z n(z')k_0dz'$$

This is the WKB approximation
(Wentzel, Kramers, Brillouin)

The electric and magnetic fields are now

$$E_x = \frac{E_0}{\sqrt{n}}\exp[i(k_0\int_0^z ndz' - \omega t)]$$
$$H_y = \sqrt{n}H_0\exp[i(k_0\int_0^z ndz' - \omega t)]$$

Now the Poynting vector is constant: $\mathbf{S} = (\mathbf{E}\times\mathbf{H}^*)/2 = (E_0H_0/2)\,\mathbf{e}_z$

This implies that no net energy is carried by the partially reflected waves due to the interference of different phases of waves reflected at different layers. This requires that the profile is smooth enough as compared to the wavength of the wave. Let's try to quantify this.

Amplitude of each partial wave is $(\triangle n/2n)(E_0/\sqrt{n})$ and the phase difference between waves reflected from two consecutive layers $2nk_0\triangle z$

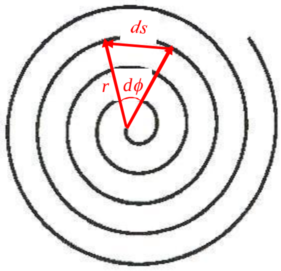
Construct a phase amplitude diagram with

$$\triangle s = \frac{\triangle n}{2n}\frac{E_0}{\sqrt{n}} \; ; \; \triangle\phi = 2nk_0\triangle z$$

arc element representing each partial wave → $\triangle s$

associated phase → $\triangle\phi$

Add arc elements to each other by turning the next one with respect to the previous one by the phase angle



Move to the continuous limit

$\triangle z \rightarrow dz,\ \triangle n \rightarrow dn,\ \triangle s \rightarrow ds,\ \triangle\phi \rightarrow d\phi$

If no reflection would take place, this would give a circle. In case of reflection the radius increases resulting in a spiral

$$\triangle r = \lim \frac{\triangle s}{\triangle \phi} = \frac{ds}{d\phi} = \frac{E_0}{4n^{5/2}k_0}\frac{dn}{dz}$$

The increase of the radius can be neglected if $\frac{1}{4n^{5/2}k_0}\left|\frac{dn}{dz}\right| \ll 1$

Thus the WKB approximation breaks if
- k_0 is small (long wave wrt. the gradient scale length
- $n \approx 0$ (close to the actual reflection point)
- at local gradients with otherwise smooth profile

Above the reflection region $n^2 < 0$ and thus n is imaginary

$$E_x = \frac{E_0}{\sqrt{n}}\exp(-i\omega t)\exp(-k|n|z)\ ;\ H_y = i|n|E_x$$

the wave is damped

At the reflection point $n \rightarrow 0 \ \Rightarrow\ E_0 \rightarrow 0$ the WKB approximation breaks

The field can, can however, be calculated analytically if the profile close to the reflection point is linear.

Now Maxwell's equations can be written as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ \Rightarrow\ \frac{dE_x}{dz} = \mu_0 i\omega H_y$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \ \Rightarrow\ \frac{dH_y}{dz} = n^2 i\omega\epsilon_0 E_x$$

$$\Rightarrow\ \frac{d^2E_x}{dz^2} + k_0^2 n^2 E_x = 0 \ \Rightarrow\ \frac{d^2E_x}{dz^2} + k_0^2\left(\frac{z_0 - z}{L}\right)E_x = 0$$

assuming linear profile

change of variable ⇒

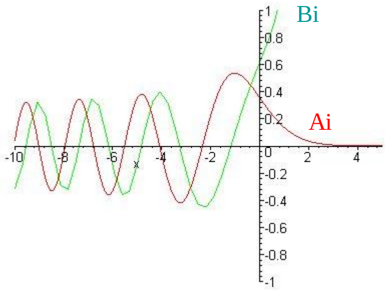
$$\frac{d^2E_x}{d\zeta^2} - \zeta E_x = 0$$

$$\frac{d^2E_x}{d\zeta^2} - \zeta E_x = 0$$ is known as Airy's differential equation

whose solutions are $E_x(\zeta) = C_1\mathrm{Ai}(\zeta) + C_2\mathrm{Bi}(\zeta)$

$$\begin{aligned}\mathrm{Ai}(\zeta) &\approx \frac{1}{2\sqrt{\pi}}\zeta^{-1/4}\exp(-\frac{2}{3}\zeta^{3/2}) \stackrel{\zeta\to\infty}{\longrightarrow} 0 \\ \mathrm{Bi}(\zeta) &\approx \frac{1}{\sqrt{\pi}}\zeta^{-1/4}\exp(\frac{2}{3}\zeta^{3/2}) \stackrel{\zeta\to\infty}{\longrightarrow} \infty\end{aligned}$$

Airy integrals



As the wave must vanish above the reflection point, $C_2 = 0$

The amplitude and period increase when $\zeta \to 0^-$

Thereafter the field approaches rapidly 0

Ai can be given in the integral form $\mathrm{Ai}(\zeta) = \frac{1}{\pi}\int_0^\infty \cos(\zeta s + \frac{s^3}{s})\, ds$

To determine the constant C_1 the Airy function solution must join the WKB solution at large negative ζ. Finding the asymptotic behaviour of Ai for negative argument is technically difficult (roots of negative numbers). The result is

$$\mathrm{Ai}(\zeta) \approx \frac{1}{2\sqrt{\pi}}\zeta^{-1/4}\left(\exp(-\frac{2}{3}\zeta^{3/2}) + i\exp(\frac{2}{3}\zeta^{3/2})\right)$$

Matching this with the WKB solution gives $C_1 = 2\sqrt{\pi}E_0(k_0L)^{1/6}$

$$\begin{aligned}E_x &= \frac{2E_0}{\sqrt{n}}\cos\left(k_0\int_z^{z_0} n\,dz' + \frac{\pi}{4}\right)\exp(-i\omega t) \\ &= \frac{E_0}{\sqrt{n}}\left\{\exp\left[\frac{i\pi}{4} + i\left(k_0\int_z^{z0} n\,dz' - \omega t\right)\right]\right. \\ &\quad \left. + \exp\left[\frac{-i\pi}{4} + i\left(-k_0\int_z^{z0} n\,dz' - \omega t\right)\right]\right\}\end{aligned}$$

upward propagating WKB solution

phase shift $\pi/4 + \pi/4 = \pi/2$ coming from the non-WKB region

downward propagating WKB solution

The reflection coefficient is $R = i \exp\left(2ik_0 \int_z^{z_0} n\, dz'\right)$

i comes from the phase shift

In the reflection region $E_x = 2\sqrt{\pi} E_0 (k_0 L)^{1/6} \mathrm{Ai}(\zeta) \exp(-i\omega t)$

gradient scale length

$\mathrm{Max[Ai]} \approx 0.55$

$f = 5 \text{ MHz} \Rightarrow \quad k_0 \approx 0.1 \text{ m}^{-1}$

Let $L \approx$ 100 km $\Rightarrow$ $E_{x,max} \approx 9\, E_0$ and λ increases by a factor of 14

ffor a perfect mirror: $E_{x,max} = 2\, E_0$

If the wave is strong enough, it can couple to the oscillation modes of plasma and lead to heating of the medium.

For oblique propagation substitute $n^2 \to q^2 = n^2 - \sin^2\theta_i$

Include collisions

Model the effect of collisions as friction to electron motion

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E} - m\nu\mathbf{v}$$

Harmonic time-dependence $\Rightarrow$ $\mathbf{v} = \dfrac{e\mathbf{E}_0 \exp(-i\omega t)}{m(i\omega - \nu)}$

$$\Rightarrow \quad \begin{cases} \epsilon = \left(1 - \dfrac{\omega_{pe}^2}{\omega^2(1 + i\nu/\omega)}\right)\epsilon_0 \\ k^2 = \mu_0\epsilon_0\omega^2\left(1 - \dfrac{\omega_{pe}^2}{\omega^2(1 + i\nu/\omega)}\right) \end{cases}$$

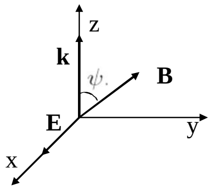
Denote $Z = \nu/\omega.$

$$\begin{cases} k^2 = k_0^2\left(1 - \dfrac{X}{1 + iZ}\right) \\ n = \sqrt{1 - \dfrac{X}{1 + iZ}} \end{cases}$$

the WKB solution is now complicated than in the loss-free case

Include background magnetic field

"magnetoionic theory"



Polar ionosphere $f_{ce} \approx 1.4\ \mathrm{MHz}$

The unmagnetized theory is limited to frequencies $f > 5\ \mathrm{MHz}$

Denote $Y = \omega_{ce}/\omega$

polarization: $\rho = \frac{1}{2}\left(-i\frac{Y\sin^2\psi/\cos\psi}{1-X-iZ} \pm 2i\sqrt{1+\frac{Y^2\sin^4\psi/\cos^2\psi}{4(1-X+iZ)}}\right)$

refractive index: $n^2 =$

$$1 - \frac{X}{1+iZ-\frac{Y^2\sin^2\psi}{2(1-X+iZ)} \mp \sqrt{Y^2\cos^2\psi+\frac{Y^4\sin^4\psi}{4(1-X+iZ)}}}$$

Appleton-Hartree equations

+ in ρ & – in n^2 : ordinary mode
– in ρ & + in n^2 : extraordinary mode

More of this with a more modern formalism in the following.

Cold plasma waves

EM wave propagation through and interaction with plasmas belong to central issues of plasma physics.
- diagnostics of the medium
  (recognize $\omega_p \rightarrow$ density, recognize $\omega_c \rightarrow$ magnetic field, and other much more complicated methods)
- understanding of the effects of the medium to the "original" signal

Meaning of cold: $v_p \gg v_{th} = \sqrt{2k_BT/m}$

Strategy: look for $\underbrace{\epsilon(\omega, \mathbf{k})}_{\text{requires physics understanding}}$ and find its zeros $\rightarrow$ $\underbrace{\text{dispersion equation}}_{\text{solutions often require numerical methods}}$

Cold plasma dispersion equation

Starting from Maxwell's equations and Ohm's law (where σ is a tensor) we can derive the wave equation

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \frac{\omega^2}{c^2} \mathcal{K} \cdot \mathbf{E} = 0$$

where $\mathcal{K} = \mathcal{I} + \frac{i}{\omega \epsilon_0} \sigma$

The dielectric tensor $\mathcal{K}$ is a dimensionless quantity: $\mathbf{D} = \epsilon \cdot \mathbf{E} = \epsilon_0 \mathcal{K} \cdot \mathbf{E}$

If there are no background fields $(\mathbf{E}_0 = \mathbf{B}_0 = 0)$
this reduces to $K = 1 - \omega_p^2/\omega^2$

In the absence of $\mathbf{B}_0$ we already know two solutions of the wave equation:

$$\begin{array}{lll} \mathbf{k} \parallel \mathbf{E} & \Rightarrow \quad \omega^2 = \omega_p^2 & \text{plasma oscillation} \\ \mathbf{k} \perp \mathbf{E} & \Rightarrow \quad \omega^2 = k^2 c^2 + \omega_p^2 & \text{electromagnetic wave} \end{array}$$

Assume then that the plasma is in a homogeneous background $\mathbf{B}_0$ and consider small perturbations: $\mathbf{B}_1 \ (B_1 \ll B_0)$

Now we also include ions, thus the current is: $\mathbf{J} = \sum_\alpha n_\alpha q_\alpha \mathbf{V}_\alpha$

Note: As plasma is assumed cold, the average velocity is the same as the velocity of individual particles

Assuming harmonic time dependence of $\mathbf{V}$: $\mathbf{V} \propto \exp(-i\omega t)$
the equation of motion for each species is

$$-i\omega m_\alpha \mathbf{V}_\alpha = q_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}_0)$$

This equation is easiest to study in the coordinate system with base vectors
$\{\sqrt{1/2}(\mathbf{e}_x + i\mathbf{e}_y), \sqrt{1/2}(\mathbf{e}_x - i\mathbf{e}_y), \mathbf{e}_z\}$ where $\mathbf{B}_0 \parallel \mathbf{e}_z$

That is, the plane perpendicular to $\mathbf{B}_0$ is considered as a complex plane (recall the discussion of wave polarization)

Let $d = \{-1, 1, 0\}$ index the components in this frame.

Introduce the shorthand notation $X_\alpha = \frac{\omega_{p\alpha}^2}{\omega^2}$, $Y_\alpha = \frac{s_\alpha \omega_{c\alpha}}{\omega}$

(here $\omega_{c\alpha}$ is positive and s_α is the sign of the charge of the species)

Now the components of the current $\mathbf{J} = \sum_\alpha n_\alpha q_\alpha \mathbf{V}_\alpha$
are $J_{d,\alpha} = i\epsilon_0 \omega \frac{X_\alpha}{1 - dY_\alpha} E_d$

and the dielectric tensor $\mathcal{K} = \mathcal{I} + \frac{i}{\omega \epsilon_0} \sigma$ in this frame is

$$\mathcal{K} = \begin{bmatrix} 1-\sum_\alpha \frac{X_\alpha}{1-Y_\alpha} & 0 & 0 \\ 0 & 1-\sum_\alpha \frac{X_\alpha}{1+Y_\alpha} & 0 \\ 0 & 0 & 1-\sum_\alpha X_\alpha \end{bmatrix}$$

The elements of the tensor can be written as

$$R = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\left(\frac{\omega}{\omega + s_\alpha \omega_{c\alpha}}\right)$$

The denominator of R is zero when the wave frequency equals to electron gyro frequency: Resonance with electrons

$$L = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\left(\frac{\omega}{\omega - s_\alpha \omega_{c\alpha}}\right)$$

The denominator of L is zero when the wave frequency equals to positive ion gyro frequency: Resonance with ions

$$P = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}.$$

Plasma oscillation

R and L refer to right-hand and left-hand polarized waves when $\mathbf{k} \parallel \mathbf{B}_0$

$\mathcal{K}$ can be transformed back to the base $\{x, y, z\}$

$$\Rightarrow \quad \mathcal{K} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

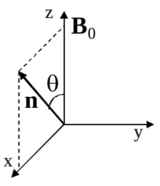
where $S = (R+L)/2$, $D = (R-L)/2$

Consider the index of refraction as a vector $\mathbf{n} = c\mathbf{k}/\omega$

Thus the wave equation can be written as

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) + \mathcal{K} \cdot \mathbf{E} = 0$$

Choose coordinates as



$$\Rightarrow$$

$$\begin{bmatrix} S - n^2\cos^2\theta & -iD & n^2\cos\theta\sin\theta \\ iD & S - n^2 & 0 \\ n^2\cos\theta\sin\theta & 0 & P - n^2\sin^2\theta \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

The non-trivial solutions of this wave equation are given by

$$An^4 - Bn^2 + C = 0$$

(this is the dispersion equation!)

where

$$A = S\sin^2\theta + P\cos^2\theta$$
$$B = RL\sin^2\theta + PS(1+\cos^2\theta)$$
$$C = PRL.$$

This is a quite convenient formulation, as it readily allows for consideration of propagation to different directions with respect to background magnetic field.

Solving for $\tan^2\theta$ we get

$$\tan^2\theta = \frac{-P(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)}$$

Principal modes: $\theta = 0$ $\theta = \pi/2$

The principal mode solutions for $\tan^2\theta = \dfrac{-P(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)}$ are

$$\theta = 0: \quad P = 0,\ n^2 = R,\ n^2 = L$$
$$\theta = \pi/2: \quad n^2 = RL/S,\ n^2 = P\,.$$

Note that $\theta = \pi/2$ refers to the angle between $\mathbf{k}$ and $\mathbf{B}_0$!

For EM-wave $\mathbf{k} \perp \mathbf{B}_1$

Cut-off

$$n^2 \to 0 \quad (v_p \to \infty,\ k \to 0,\ \lambda \to \infty)$$
$$P = 0,\ R = 0,\ \text{or}\ L = 0$$

The wave cannot propagate through the cut-off (reflection)

Resonance

$$n^2 \to \infty \quad (v_p \to 0,\ k \to \infty,\ \lambda \to 0)$$
$$\tan^2\theta = -P/S \ (\text{under the condition}\ P \neq 0)$$

Wave energy is absorbed by the plasma (acceleration, heating)

Examples of resonances for parallel propagation ($\theta = 0$):

$R \to \infty$ resonance with right-hand polarized wave (electrons)

$L \to \infty$ resonance with left-hand polarized wave (positive ions)

Wave modes in cold plasma

Parallel propagation $\theta = 0$

(consider here and below one positive ion species)

The right-hand polarized mode:

$$n_R^2 = R = 1 - \frac{\omega_{pi}^2}{\omega(\omega+\omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega-\omega_{ce})}$$

resonance with electron gyro motion

Cut-off: $\omega_R \approx \dfrac{\omega_{ce}}{2}\left[1 + \sqrt{1 + 4\omega_{pe}^2/\omega_{ce}^2}\right]$

small n limit $\omega_R \approx \omega_{ce}(1 + \omega_{pe}^2/\omega_{ce}^2)$

large n limit $\omega_R \approx \omega_{pe} + \omega_{ce}/2$

At low frequency limit $\omega \to 0$ $n^2 \to c^2/v_A^2$ (Alfvén wave)

At high frequency limit $\omega \to \infty$ $n^2 \to 1 - \omega_{pe}^2/\omega^2$ (EM wave)

The left-hand polarized mode:

$$n_L^2 = L = 1 - \frac{\omega_{pi}^2}{\omega(\omega-\omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega+\omega_{ce})}$$

resonance with ion gyro motion

Cut-off:

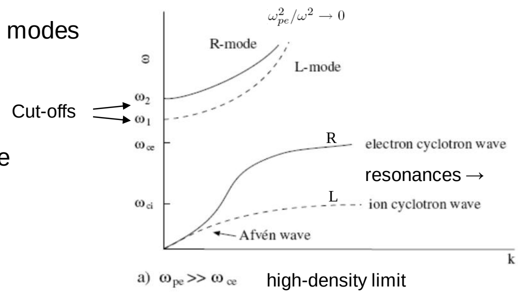
small n limit $\omega_L \approx \omega_{ci}(1 + \omega_{pi}^2/\omega_{ci}^2)$

large n limit $\omega_L \approx \omega_{pe} - \omega_{ce}/2.$

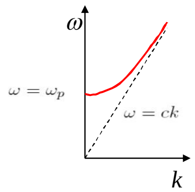
Small and large density limits mean $\omega_{pe} << \omega_{ce}$ and $\omega_{pe} >> \omega_{ce}$

In non-magnetized plasma the cut-off was at ω_p , now it is split to two branches.

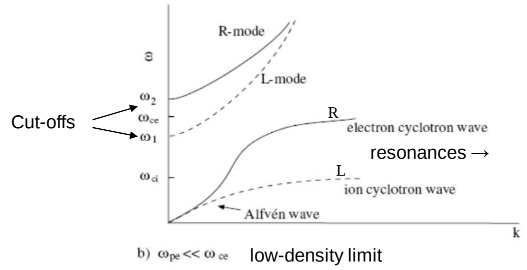
Parallel propagating wave modes in (ω, k)-space



Note the differences to the non-magnetized case:



The EM-branch is split to left- and right-handed components that have different phase velocities!



Faraday rotation

An important consequence of the different phase velocities of the L- and R-modes is the so-called Faraday rotation. Consider a linearly polarized wave and express its electric field as a sum of L- and R-polarized components:

$$\mathbf{E} = [\mathbf{e}_x(E_R e^{ik_R z} + E_L e^{ik_L z}) + i\mathbf{e}_y(E_R e^{ik_R z} - E_L e^{ik_L z})]e^{-i\omega t}$$

For a given ω, $k_R \neq k_L$ and

$$\frac{E_x}{E_y} = -i\,\frac{1 + (E_{xL}/E_{xR})\exp[i(k_L - k_R)z]}{1 - (E_{xL}/E_{xR})\exp[i(k_L - k_R)z]}$$

For a linearly polarized wave $E_{xL} = E_{xR}$

$$\Rightarrow \qquad \frac{E_x}{E_y} = \cot\left(\frac{k_L - k_R}{2}z\right)$$

i.e., the polarization plane rotates as a function of z

The rotation angle $\phi = (k_L - k_R)z/2$ depends on n and $\mathbf{B}$

In astrophysics the plasma and gyro frequencies are small as compared to the frequency of EM waves we use in our observation. Thus the dispersion equation can be approximated as

$$k_{L,R} \approx \frac{\omega}{k}\left[1 - \frac{\omega_{pe}^2}{2\omega^2}\left(1 \pm \frac{\omega_{ce}}{\omega}\right)\right]$$

⇒ the differential rotation of the plane of polarization is

$$\frac{d\phi}{dz} = \frac{-\omega_{pe}^2 \omega_{ce}}{2c\,\omega^2} = \frac{-e^3}{2m_e^2 \epsilon_0 c\,\omega^2} n_e B_0$$

Integrate this from the source to the observer:

$$\phi = \frac{-e^3}{2m_e^2 \epsilon_0 c\,\omega^2} \int_0^d n_e \mathbf{B} \cdot d\mathbf{s}$$

In astrophysics the rotation measure *RM* is defined by

$$\phi = -(RM) f^{-2}$$

Numerically

$$(RM) = 23.5 \int_0^d n_e \mathbf{B} \cdot d\mathbf{s}$$

(f: Hz; n_e: $\mathrm{cm^{-3}}$; $\mathbf{B}$: nT; $d\mathbf{s}$: m)

Beware of units in the literature!!

Note that the rotation is determined modulo π
→ observations at several frequencies needed

Dispersion measure

Assume a distant broad-band source (e.g., a pulsar) at frequencies higher than plasma and gyro frequencies. From the dispersion equation:

$$\frac{1}{v_g} \approx \frac{1}{c}\left(1 + \frac{\omega_{pe}^2}{2\omega^2}\right)$$

The propagation time of the signal to the observer is

$$T = \int_0^d \frac{ds}{v_g} = \frac{d}{c} + \frac{1}{2c\,\omega^2}\int_0^d \omega_{pe}^2 ds = \frac{d}{c} + D f^{-2}$$

D is the dispersion measure $\quad D = \frac{e^2}{8\pi^2 \epsilon_0 m_e c}\int_0^d n_e\, ds$

$$D(\mathrm{MHz}) = 4.07 \cdot 10^3 \int_0^d n_e(\mathrm{cm}^{-3})\, ds(\mathrm{pc}) \longleftarrow 1\ \mathrm{pc} = 3.09 \cdot 10^{16}\ \mathrm{m}$$

Determining the time delay between two frequencies gives average density along the propagation path $\quad \Delta T = D\left(\frac{1}{f_2^2} - \frac{1}{f_1^2}\right)$

Example:

1. Determine the distance from the redshift
2. Estimate the density from $D \rightarrow n_e$
3. Using n_e and Faraday rotation, estimate $\mathbf{B}$

Whistler mode

The R-mode propagates also in the frequency range $\omega_{ci} \ll \omega \ll \omega_{ce}$
where the dispresion equation can be approximated as

$$k = \frac{\omega_{pe}}{c}\sqrt{\frac{\omega}{\omega_{ce}}} \quad \Rightarrow \quad v_p = \frac{\omega}{k} = \frac{c\sqrt{\omega_{ce}}}{\omega_{pe}}\sqrt{\omega}$$

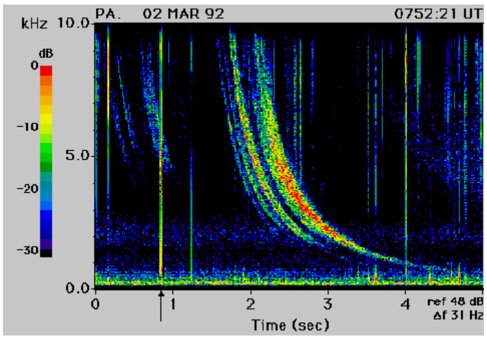
$$v_g = \frac{\partial \omega}{\partial k} = \frac{2c\sqrt{\omega_{ce}}}{\omega_{pe}}\sqrt{\omega}$$

From the group velocity we can calculate the propagation time (as a function of frequency)

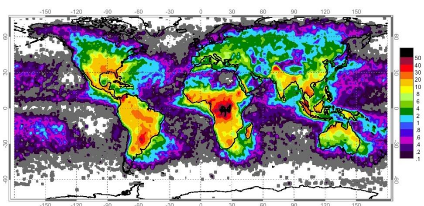
$$t(\omega) = \int \frac{ds}{v_g} = \int \frac{\omega_{pe}(s)}{2c\sqrt{\omega\omega_{ce}}}\, ds$$

Thus the lower frequencies arrive to an observer after a longer time than the higher frequences.

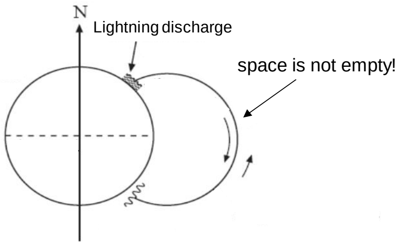
Easily observable with a simple antenna and audio amplifier (kHz frequency range)

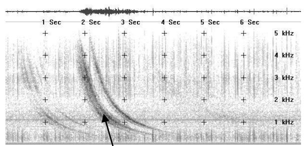


The whistlers were first observed on telegraph lines during World War I. The phenomenon was not explained until 1953, when L. R. O. Storey described the phenomenon to be due to lightning strokes on the conjugate hemisphere and propagating along the magnetic field to the observer.



Number of lightning strokes / km^2 / year
There are 16 million lightning storms per year!
Thus there are whistlers in space all the time.





Two-hop whistler

Perpendicular propagation

Modes propagating perpendicular to the background magnetic field $\theta = \pi/2$ are called ordinary (O) and extraordinary (X) modes. Unfortunately the convention in plasma physics is opposite to optics!

O-mode: $n_O^2 = P = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2} \approx 1 - \frac{\omega_{pe}^2}{\omega^2}$

This corresponds to the EM mode in isotropic plasma with cut-off at $\omega = \omega_{pe}$ (to remember: B is not included in dispersion eq. of the O-mode)

X-mode: $n_X^2 = RL/S$ (in the following we make use of $\omega_{ce} \gg \omega_{ci}$)

The X-mode has two hybrid resonances

The upper hybrid resonance: $\omega_{UH}^2 \approx \omega_{pe}^2 + \omega_{ce}^2$

The lower hybrid resonance: $\omega_{LH}^2 \approx \frac{\omega_{ci}^2 + \omega_{pi}^2}{1 + (\omega_{pe}^2/\omega_{ce}^2)} \approx \omega_{ce}\omega_{ci}\left(\frac{\omega_{pe}^2 + \omega_{ce}\omega_{ci}}{\omega_{pe}^2 + \omega_{ce}^2}\right)$

The lower hybrid resonance is particulary important because there the wave can be in a resonance of both electrons and ions, which provides efficient means for energy transfer between particle populations

Low-density limit: $\omega_{LH} \to \omega_{ci}$ High-density limit: $\omega_{LH} \to \sqrt{\omega_{ce}\omega_{ci}}$

X-mode has also two cut-offs:

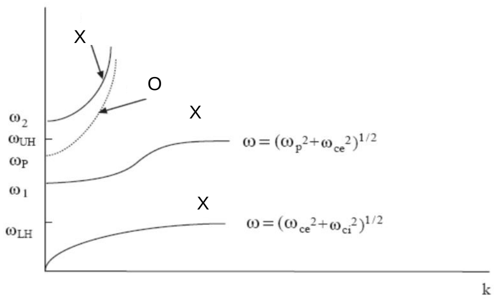
Low-density limit $\begin{cases} \omega_X = \omega_{ce} + \frac{\omega_{pe}^2}{\omega_{ce}} \\ \omega_X = \omega_{ci} + \frac{\omega_{pe}^2}{\omega_{ce}} \end{cases}$ High-density limit: $\omega_X = \omega_{pe} \pm \frac{1}{2}\omega_{ce}$

When $\omega \to 0$ the X-mode approaches the magnetosonic mode of MHD

$$n_X^2 \to 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} = 1 + \frac{c^2}{v_A^2}$$

$$\frac{\omega^2}{k^2} = v_s^2 + v_A^2$$

In MHD finite T assumed (not in cold theory)



Effect of both finite T and displacement current

$$\frac{\omega^2}{k^2} = \frac{v_s^2 + v_A^2}{1 + v_A^2/c^2}$$

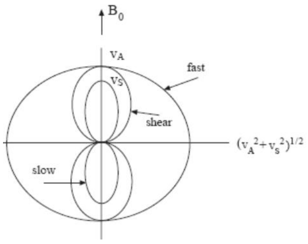
correction due to the displacement current; v_A can be a considerable fraction of c

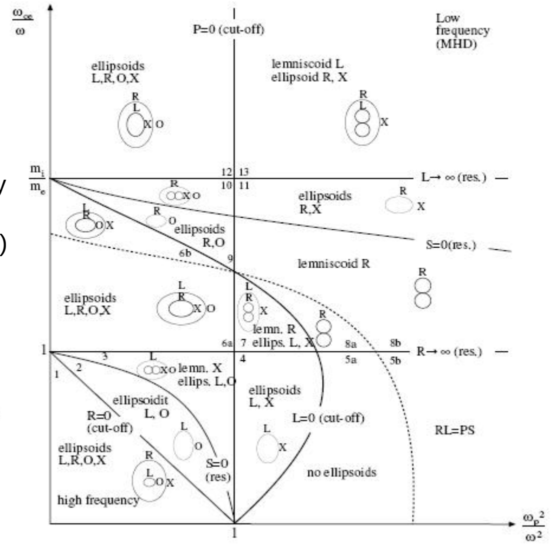
Propagation at arbitrary angles

Clemmow-Mullaly-Allis (CMA) diagram
Topologies of wave-normal surfaces
B is upward

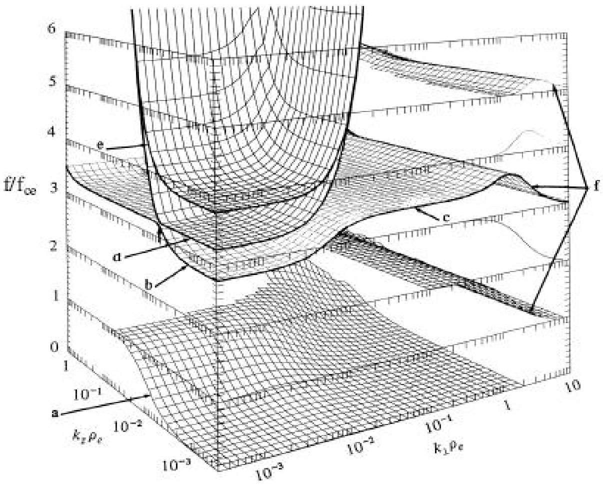
The R, L, O, X division is unambiguous for principal modes only ($\theta = 0, \pi/2$).

Wave normal surfaces:
- calculate the phase velocity for different θ:
- e.g., MHD waves (next Ch.)





Presentation of wave modes in $(\omega, k_{\|}, k_{\perp})$ space
(here electron-related modes only)



a) Whistler
b) L-mode
c) Upper hybrid mode
d) Plasma frequency (Langmuir wave)
e) R-mode
f) Bernstein modes (not discussed here)

This picture has been computed in Vlasov theory and thus is more exact than the cold and MHD approximations

Excercise: Identify O- and X-modes, where do the thermal effects on Langmuir wave start to play role, etc. ?