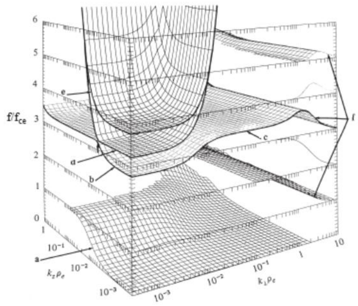
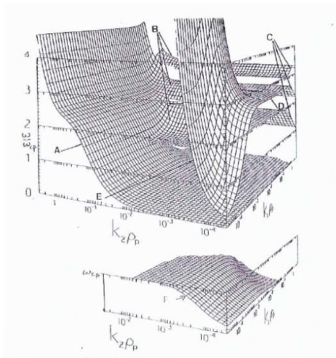
Vlasov theory

How to solve

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0}$$





VE $\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = 0$ **conserves particles**:

Integrate over the $(\mathbf{r},\mathbf{v})$-space $\Rightarrow$ $\frac{\partial}{\partial t}\int n_\alpha f_\alpha d^3r d^3v = 0$

Positive probabilities remain positive in the Vlasov description:

If $f_\alpha(\mathbf{r},\mathbf{v},t=0) > 0$ for all $(\mathbf{r},\mathbf{v})$, then $f_\alpha(\mathbf{r},\mathbf{v},t) > 0$ for all $t > 0$

VE has many equilibrium solutions

Boltzmann's H-theorem: In the collisional time scale
the Maxwellian distribution is a unique solution. But for us $\partial f/\partial t|_c = 0$

Let $f_{\alpha 0}$ be any Vlasov equilibrium, then $\partial f_{\alpha 0}/\partial t = 0$ and thus

$$\left[\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial}{\partial \mathbf{v}}\right] f_{\alpha 0} = 0 \qquad (*)$$

Consider an orbit $(\mathbf{r}'(t'),\mathbf{v}'(t'))$ that intersects the point $(\mathbf{r},\mathbf{v})$ at time $t' = t$

If $a(\mathbf{r}',\mathbf{v}'),\ b(\mathbf{r}',\mathbf{v}'), \ldots$ are constants of motion, then any function

$f_{\alpha 0}[a(\mathbf{r}',\mathbf{v}'),\ b(\mathbf{r}',\mathbf{v}'), \ldots]$ satisfies (*) at time $t' = t$

Thus any function $f_{\alpha 0}[a(\mathbf{r},\mathbf{v}),\ b(\mathbf{r},\mathbf{v}), \ldots]$ is a stationary-state solution of VE

Examples:

$\mathbf{E}_0 = \mathbf{B}_0 = 0$

Constants of motion are

$$\begin{aligned} W &= \frac{m_\alpha}{2}(v_x^2 + v_y^2 + v_z^2) \\ \mathbf{p} &= m_\alpha \mathbf{v}\,. \end{aligned}$$

With these we can construct, for example, the following equilibrium distributions

$$\begin{aligned} f_{\alpha 0} &= \left(\frac{m_\alpha}{2\pi k_B T_\alpha}\right)^{3/2} \exp\left(-\frac{m_\alpha}{2 k_B T_\alpha} v^2\right) \\ f_{\alpha 0} &= \frac{v_0}{2}\frac{1}{v^4 + v_0^4} \\ f_{\alpha 0} &= v_0\,\delta(v_x)\,\delta(v_y)\,\delta(v_z^2 - v_0^2) \\ f_{\alpha 0} &= \sqrt{\frac{m_\alpha}{2\pi k_B T_\alpha}}\,\delta(v_x)\,\delta(v_y)\,\exp\left(-\frac{m_\alpha(v_z^2 - v_{\alpha 0}^2)}{2 k_B T_\alpha}\right) \end{aligned}$$

$\mathbf{E}_0 = 0,\ \mathbf{B}_0 = B_0(\mathbf{r})\,\mathbf{e}_z$

One set of constants of motion:

$$\begin{aligned} W &= \frac{m_\alpha}{2}(v_x^2 + v_y^2 + v_z^2) \\ p_\parallel &= m_\alpha v_z \\ \mathbf{L} &= m_\alpha(x v_y - y v_x)\,\mathbf{e}_z - q_\alpha r A_\phi(r)\,\mathbf{e}_\phi \end{aligned}$$

or another:

$$\begin{aligned} \xi_x &= v_x - \frac{q_\alpha}{m_\alpha}\int B_0(r)\,dy \\ \xi_y &= v_y + \frac{q_\alpha}{m_\alpha}\int B_0(r)\,dx \end{aligned}$$

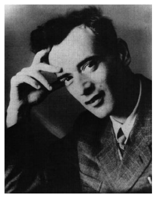
One of many equilibria for constant B_0 is

$$f_{\alpha 0} = F\left(v^2,\ v_y + \frac{q_\alpha B_0}{m_\alpha}x,\ v_x - \frac{q_\alpha B_0}{m_\alpha}y\right)$$

VE conserves entropy $S = -\sum_\alpha \int f_\alpha \ln f_\alpha\, d^3r d^3v$

$$\frac{dS}{dt} = -\sum_\alpha \int \left(\frac{df_\alpha}{dt}\ln f_\alpha + \frac{df_\alpha}{dt}\right) d^3r d^3v = 0$$

Landau's solution of VE


Lev Landau

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

Very hard task in a general case.
VE is nonlinear, thus we linearize:

$$\begin{aligned} f_\alpha &= f_{\alpha 0} + f_{\alpha 1} \\ \mathbf{E} &= \mathbf{E}_0 + \mathbf{E}_1 \\ \mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1 \end{aligned}$$

Consider a homogeneous, field-free plasma $\mathbf{E}_0 = \mathbf{B}_0 = 0$
in an electrostatic approximation: $\mathbf{E}_1 = -\nabla \varphi_1; \ \mathbf{B}_1 = 0$

The linearized VE is now $\frac{\partial f_{\alpha 1}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\alpha 1}}{\partial \mathbf{r}} - \frac{q_\alpha}{m_\alpha} \frac{\partial \varphi_1}{\partial \mathbf{r}} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}} = 0$

where $\nabla^2 \varphi_1 = -\frac{1}{\epsilon_0} \sum_\alpha n_\alpha q_\alpha \int f_{\alpha 1} \, d^3 v$

Vlasov tried this at the end of the 1930s using Fourier transformation

$\rightarrow$ an integral of the form $\int_{-\infty}^{\infty} \frac{\partial f_{\alpha 0}/\partial v}{\omega - kv} \, dv$

pole along the integration path, what to do?

Landau's physical intuition: The perturbation must appear at some instant
$\rightarrow$ consider this as an initial value problem in time, make Laplace-transformation, and look for time asymptotic solutions

$$f_{\alpha \mathbf{k}}(\mathbf{v}, t) = \frac{1}{(2\pi)^3} \int f_{\alpha 1}(\mathbf{r}, \mathbf{v}, t) \exp(-i\mathbf{k} \cdot \mathbf{r}) d^3 r$$

Fourier transform in space

$$\tilde{f}_{\alpha \mathbf{k}}(\mathbf{v}, p) = \int_0^\infty f_{\alpha \mathbf{k}}(\mathbf{v}, t) \exp(-pt) dt \quad ; \quad Re(p) \geq p_0$$

Laplace transform in time

and the same for $\varphi(\mathbf{r}, t)$

Solve the algebraic eqs for $\tilde{f}_{\alpha \mathbf{k}}$ and $\tilde{\varphi}_{\mathbf{k}}$ and take the inverse transformations:

$$\begin{aligned} f_{\alpha 1}(\mathbf{r}, \mathbf{v}, t) &= \int \exp(i\mathbf{k} \cdot \mathbf{r}) d\mathbf{k} \int_{p_0 - i\infty}^{p_0 + i\infty} \exp(pt) \tilde{f}_{\alpha \mathbf{k}}(\mathbf{v}, p) \frac{dp}{2\pi i} \\ \varphi_1(\mathbf{r}, t) &= \int \exp(i\mathbf{k} \cdot \mathbf{r}) d\mathbf{k} \int_{p_0 - i\infty}^{p_0 + i\infty} \exp(pt) \tilde{\varphi}_{\mathbf{k}}(p) \frac{dp}{2\pi i} . \end{aligned}$$

The transformed equations are

$$
\begin{aligned}
(p + i\mathbf{k}\cdot\mathbf{v})\tilde{f}_{\alpha\mathbf{k}} &= f_{\alpha\mathbf{k}}(\mathbf{v}, t=0) + \frac{q_\alpha}{m_\alpha}\left(i\mathbf{k}\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}\right)\tilde{\varphi}_{\mathbf{k}} \\
k^2\tilde{\varphi}_{\mathbf{k}} &= \frac{1}{\epsilon_0}\sum_\alpha n_\alpha q_\alpha \int \tilde{f}_{\alpha\mathbf{k}}\, d^3v\,.
\end{aligned}
$$

Solve the potential:

$$
k^2\tilde{\varphi}_{\mathbf{k}} = \frac{\dfrac{1}{\epsilon_0}\displaystyle\sum_\alpha n_\alpha q_\alpha \int \frac{f_{\alpha\mathbf{k}}(t=0)}{p + i\mathbf{k}\cdot\mathbf{v}} d^3v}{\boxed{1 + \dfrac{1}{\epsilon_0}\displaystyle\sum_\alpha \frac{n_\alpha q_\alpha^2}{m_\alpha}\frac{1}{k^2}\int \frac{\mathbf{k}\cdot\partial f_{\alpha 0}/\partial\mathbf{v}}{ip - \mathbf{k}\cdot\mathbf{v}} d^3v}} \;;\quad Re(p) \geq p_0
$$

$\omega = ip$ $\qquad\qquad$ $K(\mathbf{k}, \omega)$

$$
K(\mathbf{k}, \omega) = 1 + \frac{1}{\epsilon_0}\sum_\alpha \frac{n_\alpha q_\alpha^2}{m_\alpha}\frac{1}{k^2}\int \frac{\mathbf{k}\cdot\partial f_{\alpha 0}/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}\, d^3v
$$

contains all information we usually are interested in, thus we do not need to calculate the inverse transformations in practice, but we must know how it should be done in order to calculate the integral in K

Make the integral one-dimensional:

$$
\begin{aligned}
F_{\alpha 0}(u) &\equiv \int f_{\alpha 0}(\mathbf{v})\delta\left(u - \frac{\mathbf{k}\cdot\mathbf{v}}{|k|}\right) d^3v \\
\tilde{F}_{\alpha\mathbf{k}}(u) &\equiv \int \tilde{f}_{\alpha\mathbf{k}}(\mathbf{v})\delta\left(u - \frac{\mathbf{k}\cdot\mathbf{v}}{|k|}\right) d^3v
\end{aligned}
$$

$$
\Rightarrow \quad K(\mathbf{k}, ip) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2}\int \frac{\partial F_{\alpha 0}(u)/\partial u}{u - ip/|k|} du \;;\quad Re(p) \geq p_0
$$

The inverse transform of $\tilde{\varphi}_{\mathbf{k}}$

$$
k^2\varphi_{\mathbf{k}}(t) = \int_{p_0 - i\infty}^{p_0 + i\infty} \frac{\dfrac{1}{\epsilon_0}\displaystyle\sum_\alpha n_\alpha q_\alpha \int \frac{F_{\alpha\mathbf{k}}(u, t=0)}{p + i|k|u} du}{K(\mathbf{k}, ip)} \exp(pt)\, \frac{dp}{2\pi i}
$$

can be calculated in a closed form only for some specific equilibria $F_{\alpha 0}$ and initial perturbations $F_{\alpha\mathbf{k}}(u, t=0)$

Landau: Look for asymptotic behavior when $t \to \infty$ i.e., when the initial transients have disappeared and the normal modes of the plasma remain

In order to integrate

$$k^2\varphi_{\mathbf{k}}(t) = \int\limits_{p_0-i\infty}^{p_0+i\infty} \frac{\frac{1}{\epsilon_0}\sum_\alpha n_\alpha q_\alpha \int \frac{F_{\alpha\mathbf{k}}(u,t=0)}{p+i|k|u}du}{K(\mathbf{k},ip)} \exp(pt)\,\frac{dp}{2\pi i}$$

we consider the analytic properties of $\tilde{\varphi}_{\mathbf{k}}(p)$

By definition it is analytical when $Re(p) \geq p_0$

In order to apply residue calculus make an analytic continuation of $\tilde{\varphi}_{\mathbf{k}}(p)$ to the entire *p*-plane, but how to continue

$$h(p) = \int\limits_{-\infty}^{+\infty} \frac{g(u)}{u - ip/|k|}du \quad ; \quad Re(p) \geq p_0$$

to $Re(p) < p_0$?

Assume that $g(u)$ is analytic when $|u| < \infty$

If $Re(p) > 0$ the pole of the integrand is above the integration path

The analytic continuation requires (exercise) that the integration contour Passes below the pole also for $Re(p) \leq 0$:

$$h(p) = \begin{cases} \int\limits_{-\infty}^{+\infty} \frac{g(u)du}{u-ip/|k|} & ;\ Re(p) > 0 \\ P\int\limits_{-\infty}^{+\infty} \frac{g(u)du}{u-ip/|k|} + \pi i\, g(ip/|k|) & ;\ Re(p) = 0 \\ \int\limits_{-\infty}^{+\infty} \frac{g(u)du}{u-ip/|k|} + 2\pi i\, g(ip/|k|) & ;\ Re(p) \leq 0\,, \end{cases}$$

Cauchy principal value

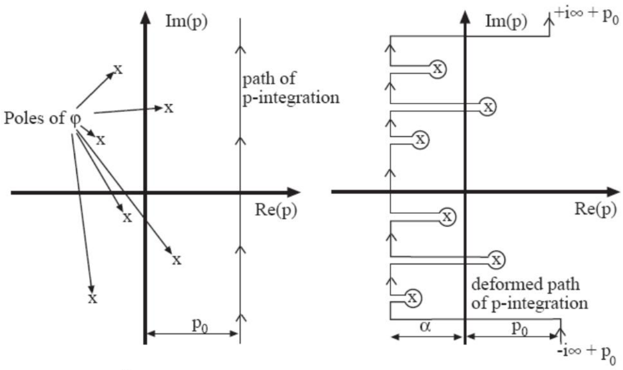
This integration path is called the Landau contour and denoted by $\int_L$

The only singularities in

$$k^2\varphi_{\mathbf{k}}(t) = \int\limits_{p_0-i\infty}^{p_0+i\infty} \frac{\frac{1}{\epsilon_0}\sum_\alpha n_\alpha q_\alpha \int \frac{F_{\alpha\mathbf{k}}(u,t=0)}{p+i|k|u}du}{K(\mathbf{k},ip)} \exp(pt)\,\frac{dp}{2\pi i}$$

are the zeros of $K(\mathbf{k},ip)$

In order to calculate the p-integral move the integration path $(-i\infty \rightarrow i\infty)$ so far to negative $Re(p)$ that $\exp(pt)$ guarantees the contribution of the vertical parts of the integration to vanish and the only thing that is left are the residues at the poles



$$\varphi_{\mathbf{k}}(t) = \sum_j R_j \exp(p_j(\mathbf{k})t) + \underbrace{\int_{-i\infty+p_0}^{-i\infty-\alpha} \tilde{\varphi}_{\mathbf{k}}(p)\exp(pt)\frac{dp}{2\pi i}}_{\rightarrow 0}$$

$$+ \underbrace{\int_{-i\infty-\alpha}^{i\infty-\alpha} \tilde{\varphi}_{\mathbf{k}}(p)\exp(pt)\frac{dp}{2\pi i}}_{\text{Exponentially smaller than the residue term as } t\rightarrow\infty} + \underbrace{\int_{i\infty-\alpha}^{i\infty+p_0} \tilde{\varphi}_{\mathbf{k}}(p)\exp(pt)\frac{dp}{2\pi i}}_{\rightarrow 0}$$

$\tilde{\varphi}_{\mathbf{k}} \rightarrow 0$

$|p| \rightarrow \infty$

Thus the time-asymptotic solution is

$$\varphi_{\mathbf{k}}(t\rightarrow\infty) = \sum_j R_j \exp(p_j(\mathbf{k})t) = \sum_j R_j \exp(-i\omega_j(\mathbf{k})t)$$

where $\omega_j = \omega_r + i\omega_i$ are the solutions of

$$K(\mathbf{k},\omega) \equiv 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2}\int_L \frac{\partial F_{\alpha 0}(u)/\partial u}{u-\omega/|k|}\,du = 0$$

$Re(p_j) < 0 \quad \Rightarrow \quad \omega_i < 0 \quad \varphi_{\mathbf{k}}$ is damped
$Re(p_j) > 0 \quad \Rightarrow \quad \omega_i > 0 \quad \varphi_{\mathbf{k}}$ grows (instability)

Normal modes: $|\omega_i| \ll |\omega_r|$

A short-cut to the same description:

Take Vlasov's approach $\Rightarrow \quad 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2}\int \frac{\mathbf{k}\cdot\partial f_{\alpha 0}/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}\,d^3v = 0$

Include weak collisions with the model $\partial f/\partial t|_c = -\nu(f - f_0)$
and let at the end $\nu \rightarrow 0+$

Normal modes in a Maxwellian plasma

Assume $\mathbf{E}_0 = \mathbf{B}_0 = 0$

$$F_{\alpha 0} = \sqrt{\frac{m_\alpha}{2\pi k_B T_\alpha}} \exp(-u^2/v_{th\alpha}^2)$$ 1-D Maxwellian

$$v_{th\alpha} = \sqrt{\frac{2k_B T_\alpha}{m_\alpha}}$$

The Landau contour is non-trivial as the integrand of

$$\int \frac{\partial F_{\alpha 0}/\partial u}{u - \omega/|k|} du \approx \int \frac{u F_{\alpha 0}}{u - \omega/|k|} du$$ diverges as $u \to \infty$

This can be solved within complex analysis and the result is expressed
In terms of the plasma dispersion function

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-x^2)}{x - \zeta} dx \ ; \ Im(\zeta) > 0$$

Considering electrons only
(ions as non-moving background)

$$1 - \frac{\omega_{pe}^2}{k^2 v_{the}^2} Z'\left(\frac{\omega}{k v_{the}}\right) = 0$$

For normal modes $|\omega_i| \ll |\omega_r|$ expand the dispersion equation around $\omega_i = 0$

$$1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2}\left(1 + i\omega_i \frac{\partial}{\partial \omega_i}\right)\left[P \int \frac{\partial F_{\alpha 0}/\partial u}{u - \omega_r/|k|} du + \pi i \left(\frac{\partial F_{\alpha 0}}{\partial u}\right)_{u=\omega_r/|k|}\right] = 0$$

Long wavelength limit $\omega/k \gg v_{th}$

$$-P \int \frac{\partial F_{\alpha 0}/\partial u}{u - \omega_r/|k|} du = \int \frac{\partial F_{\alpha 0}}{\partial u}\left(\frac{1}{\omega/|k|} + \frac{u}{(\omega/|k|)^2} + \frac{u^2}{(\omega/|k|)^3} + \ldots\right) du$$

Neglecting ions (background) and using a Maxwellian distribution for electrons:

$$\Rightarrow \begin{cases} \omega_r \approx \omega_{pe}(1 + 3k^2\lambda_{De}^2)^{1/2} \approx \omega_{pe}(1 + \frac{3}{2}k^2\lambda_{De}^2) & \text{Langmuir wave} \\ \omega_i = -\sqrt{\frac{\pi}{8}} \frac{\omega_{pe}}{|k^3\lambda_{De}^3|} \exp\left(-\frac{1}{2k^2\lambda_{De}^2} - \frac{3}{2}\right) & \text{Landau damping} \end{cases}$$

An alternative route to the same same result

Expand K around $\omega_i = 0$ $\quad K(\omega, \mathbf{k}) \approx K(\omega_r, \mathbf{k}) + i\omega_i \frac{\partial K(\omega_r, \mathbf{k})}{\partial \omega_r}$

complex function containing

$$\lim_{\epsilon \to 0^+} \int \frac{\partial F_{\alpha 0}/\partial u}{u - \omega_r/|k| - i\epsilon}\, du$$

Thus

$$\begin{aligned} K(\omega_r, \mathbf{k}) &= K_r(\omega_r, \mathbf{k}) + iK_i(\omega_r, \mathbf{k}) \\ K_i &= -\pi \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \left(\frac{\partial F_{\alpha 0}}{\partial u}\right)_{u=\omega_r/|k|} \\ K_r &= 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2}\, P \int \frac{\partial F_{\alpha 0}/\partial u}{u - \omega_r/|k|}\, du \end{aligned}$$

The Landau contour is not given explicitly, but taken care by $\epsilon \to 0^+$

Equating the imaginary parts

$\omega_i = \frac{-K_i(\omega_r, \mathbf{k})}{\partial K_r(\omega_r, \mathbf{k})/\partial \omega_r}$ where K_r fulfils the dispersion equation $K_r(\omega_r, \mathbf{k}) = 0$

The ion acoustic wave: Look for a solution when $T_e \gg T_i$

in the phase speed range $\sqrt{\frac{k_B T_i}{m_i}} < \frac{\omega}{k} < \sqrt{\frac{k_B T_e}{m_e}}$.

For ions this is the long wavelength regime but for electrons $v_{pe} < v_{the}$
and we must look for short wavelength expansion, that can be written as (exerc.)

$$P \int \frac{\partial F_{\alpha 0}/\partial u}{u - \omega_r/|k|}\, du \approx 2 \int \frac{\partial F_{\alpha 0}}{\partial (u^2)}\, du$$

Assume Maxwellian distributions for both species:

$$\begin{aligned} K_r &= 1 - \frac{\omega_{pi}^2}{\omega_r^2} + \frac{1}{k^2 \lambda_{De}^2} \\ K_i &= \pi \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \left(\frac{m_\alpha}{2\pi k_B T_\alpha}\right)^{1/2} \frac{m_\alpha}{k_B T_\alpha} \frac{\omega_r}{|k|} \exp\left(-\frac{\omega_r^2 m_\alpha}{2k^2 k_B T_\alpha}\right) \end{aligned}$$

$\omega_r^2 = \frac{k^2 c_s^2}{1 + k^2 \lambda_{De}^2}$; $c_s = \sqrt{\frac{k_B T_e}{m_i}}$ ion-acoustic wave
ion-acoustic speed

$$\begin{aligned} \omega_i &= -\frac{K_i}{\partial K_r/\partial \omega_r} \\ &= -\frac{|\omega_r|\sqrt{\pi/8}}{(1 + k^2 \lambda_{De}^2)^{3/2}} \left[\left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(\frac{-T_e/T_i}{2(1 + k^2 \lambda_{De}^2)}\right) + \sqrt{\frac{m_e}{m_i}}\right] \end{aligned}$$

the damping is very strong unless $T_e \gg T_i$

Physics of Landau damping

Landau's solution was met with scepticism:
VE conserves entropy, but damping indicates dissipation!

Take a closer look at the perturbed distribution function $f_1(t)$

$$\tilde{f}_{\alpha\mathbf{k}}(\mathbf{v},p) = \frac{f_{\alpha\mathbf{k}}(\mathbf{v},t=0)}{(p+i\mathbf{k}\cdot\mathbf{v})} + \frac{q_\alpha}{m_\alpha}\frac{i\tilde{\varphi}_{\mathbf{k}}\mathbf{k}\cdot\partial f_{\alpha 0}/\partial\mathbf{v}}{(p+i\mathbf{k}\cdot\mathbf{v})}$$

$$f_{\alpha\mathbf{k}}(\mathbf{v},t) = \frac{1}{2\pi i}\int_{p_0-i\infty}^{p_0+i\infty}\tilde{f}_{\alpha\mathbf{k}}(\mathbf{v},p)\exp(pt)\,dp\,.$$

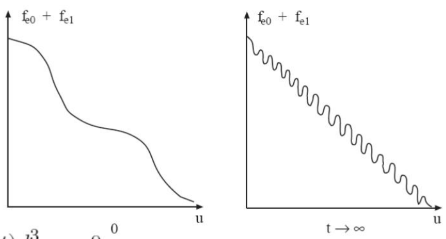
$\tilde{f}_{\alpha\mathbf{k}}(\mathbf{v},p)$ has the same poles as $\tilde{\varphi}_{\mathbf{k}}(p)$ that give the solutions of $K=0$
and a pole at $p=-i\mathbf{k}\cdot\mathbf{v}$

At the limit $t\to\infty$ $\quad f_{\alpha\mathbf{k}} = \hat{f}_{\alpha B}\exp(-i\mathbf{k}\cdot\mathbf{v}t) + \sum_{\omega_{\mathbf{k}}}\hat{f}_{\alpha\mathbf{k}}\exp(-i\omega_{\mathbf{k}}t)$

B for "ballistic", every particle remembers the initial perturbation wherever it moves in the phase space

these are damped at the same rate as $\varphi_1(t)$

When t increases the ballistic term becomes increasingly oscillatory in the velocity space and its contribution to $\varphi_1(t)$ behaves as



$$k^2\varphi_{\mathbf{k}} = \frac{1}{\epsilon_0}\sum_\alpha q_\alpha n_\alpha \int \hat{f}_{\alpha B}\exp(-i\mathbf{k}\cdot\mathbf{v}t)d^3v \underset{t\,\to\,\infty}{\to} 0$$

Thus all information is maintained, but it does not show in observable $\mathbf{E}$

This was demonstrated in laboratories in the 1960s in the following way:

Launch a perturbation at time t_1 with wavenumber $k\approx k_1$
$f_\alpha = f_{\alpha 0} + f_{\alpha\mathbf{k}1}(u,t=t_1)\exp(ik_1u(t-t_1)) + \ldots$
and let it damp away. Launch a second perturbation at t_2 with $k\approx k_2$
Add this to f_α and do not linearize!

1st order 2nd order

$$f_\alpha = f_{\alpha 0} + f_{\alpha\mathbf{k}1}^{(1)}\exp(ik_1u(t-t_1)) + f_{\alpha\mathbf{k}2}^{(1)}\exp(ik_2u(t-t_2)) + f_\alpha^{(2)} + \ldots$$

$$f_\alpha^{(2)} \approx f_{\alpha\mathbf{k}1}^{(1)}f_{\alpha\mathbf{k}2}^{(1)}\exp(ik_1u(t-t_1))\exp(-ik_2u(t-t_2))\;;\; t>t_2$$

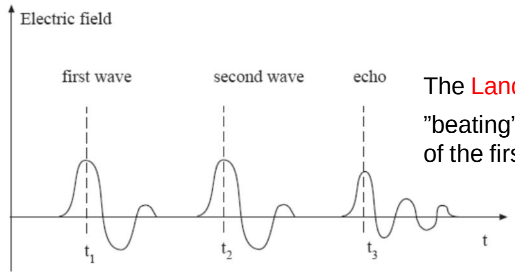
At time $t = t_3$ defined by $k_1(t_3 - t_1) - k_2(t_3 - t_2) = 0$

$$f_\alpha^{(2)} \approx f_{\alpha \mathbf{k}1}^{(1)} f_{\alpha \mathbf{k}2}^{(1)} \exp(ik_1 u(t-t_1)) \exp(-ik_2 u(t-t_2)) \; ; \; t > t_2$$

is not small.

This gives rise to an observable charge density

$$\rho_{q2} \approx \int du \exp(ik_1 u(t-t_1) - ik_2 u(t-t_2)) f_{\alpha(\mathbf{k}_2 - \mathbf{k}_1)}^{(2)}$$



The Landau echo

"beating" of the ballistic terms
of the first and second perturbation!

Thus the Landau damping does not contradict
the conservation of entropy in the time scale $\tau \ll \tau_{coll}$

Vlasov theory in general equilibrium

Inclusion of background magnetic field, inhomogeneities etc., soon lead to technical difficulties. The general method is as follows

$$\Big[\underbrace{\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial}{\partial \mathbf{v}}}_{\text{background fields}}\Big] f_{\alpha 1} = \underbrace{-\frac{q_\alpha}{m_\alpha}(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1)}_{\text{perturbed (wave) fields}} \cdot \frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}$$

Method of characteristics or integration over unperturbed orbits

Define new variables $\dfrac{d\mathbf{r}'}{dt'} = \mathbf{v}' \; ; \; \dfrac{d\mathbf{v}'}{dt'} = \dfrac{q_\alpha}{m_\alpha}\left[\mathbf{E}_0(\mathbf{r}', t') + \mathbf{v}' \times \mathbf{B}_0(\mathbf{r}', t')\right]$

with boundary conditions $\mathbf{r}'(t' = t) = \mathbf{r} \; ; \; \mathbf{v}'(t' = t) = \mathbf{v}$

Consider $f_{\alpha 1}(\mathbf{r}', \mathbf{v}', t')$

$$\frac{df_{\alpha 1}(\mathbf{r}', \mathbf{v}', t')}{dt'}$$

$$\equiv \frac{\partial f_{\alpha 1}(\mathbf{r}', \mathbf{v}', t')}{\partial t'} + \frac{d\mathbf{r}'}{dt'} \cdot \frac{\partial f_{\alpha 1}(\mathbf{r}', \mathbf{v}', t')}{\partial \mathbf{r}'} + \frac{d\mathbf{v}'}{dt'} \cdot \frac{\partial f_{\alpha 1}(\mathbf{r}', \mathbf{v}', t')}{\partial \mathbf{v}'}$$

$$= -\frac{q_\alpha}{m_\alpha}\left[\mathbf{E}_1(\mathbf{r}', t') + \mathbf{v}' \times \mathbf{B}_1(\mathbf{r}', t')\right] \cdot \frac{\partial f_{\alpha 0}(\mathbf{r}', \mathbf{v}')}{\partial \mathbf{v}'} .$$

At time $t'=t$ $\quad f_{\alpha 1}(\mathbf{r}',\mathbf{v}',t') = f_{\alpha 1}(\mathbf{r},\mathbf{v},t)$
and at that time the solution of

$$\frac{df_{\alpha 1}(\mathbf{r}',\mathbf{v}',t')}{dt'} \equiv \frac{\partial f_{\alpha 1}(\mathbf{r}',\mathbf{v}',t')}{\partial t'} + \frac{d\mathbf{r}'}{dt'}\cdot\frac{\partial f_{\alpha 1}(\mathbf{r}',\mathbf{v}',t')}{\partial \mathbf{r}'} + \frac{d\mathbf{v}'}{dt'}\cdot\frac{\partial f_{\alpha 1}(\mathbf{r}',\mathbf{v}',t')}{\partial \mathbf{v}'} = -\frac{q_\alpha}{m_\alpha}\left[\mathbf{E}_1(\mathbf{r}',t') + \mathbf{v}'\times\mathbf{B}_1(\mathbf{r}',t')\right]\cdot\frac{\partial f_{\alpha 0}(\mathbf{r}',\mathbf{v}')}{\partial \mathbf{v}'}\,.$$

is a solution of VE.
This can be calculated by a direct integration, $\dfrac{df_{\alpha 1}(\mathbf{r}',\mathbf{v}',t')}{dt'}$
because the LHS is an exact differential

$$\begin{aligned} f_{\alpha 1}(\mathbf{r},\mathbf{v},t) &= -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{t}\left[\mathbf{E}_1(\mathbf{r}',t') + \mathbf{v}'\times\mathbf{B}_1(\mathbf{r}',t')\right]\cdot\frac{\partial f_{\alpha 0}(\mathbf{r}',\mathbf{v}')}{\partial \mathbf{v}'}dt' \\ &+ f_{\alpha 1}(\mathbf{r}'(-\infty),\mathbf{v}'(-\infty),t'(-\infty))\,. \end{aligned}$$

$f_{\alpha 1}$ has been found by integrating VE from $-\infty$ to t
along the path in $(\mathbf{r},\mathbf{v})$ space that coincides with the particle orbit
in the equilibrium fields $\mathbf{E}_0$ and $\mathbf{B}_0$.
From $f_{\alpha 1}$ we calculate $n_{\alpha 1}(\mathbf{r},t)$ and $\mathbf{V}_{\alpha 1}(\mathbf{r},t)$ and insert to Maxwell's equations

$$\begin{aligned} \nabla\times\mathbf{E}_1 &= -\frac{\partial \mathbf{B}_1}{\partial t} \\ \nabla\cdot\mathbf{E}_1 &= \frac{1}{\epsilon_0}\sum_\alpha q_\alpha n_{\alpha 1} \\ \nabla\times\mathbf{B}_1 &= \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t} + \mu_0\sum_\alpha q_\alpha(n_\alpha \mathbf{V}_\alpha)_1 \end{aligned}$$

From here on we could proceed to solve an initial value problem.
Assuming that the Landau prescription is the correct way to deal with the resonant integrals and that
$\mathbf{E}_1(\mathbf{r},t) = \mathbf{E}_{\mathbf{k}\omega}\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$ & $f_{\alpha 1}(\mathbf{r}',\mathbf{v}',t\to-\infty)\to 0$
For $\mathrm{Im}(\omega) > 0$ $\qquad \tau = t'-t\,,\ \mathbf{R} = \mathbf{r}'-\mathbf{r}.$

$$f_{\alpha\mathbf{k}} = -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{0}(\mathbf{E}_{\mathbf{k}\omega} + \mathbf{v}'\times\mathbf{B}_{\mathbf{k}\omega})\cdot\frac{\partial f_{\alpha 0}(\mathbf{v}')}{\partial \mathbf{v}'}\exp[i(\mathbf{k}\cdot\mathbf{R} - \omega\tau)]d\tau$$

For $\mathrm{Im}(\omega) < 0$, continue $f_{\alpha\mathbf{k}}$ analytically to the lower half-plane
Insert this to Maxwell's equations in the $(\omega,\mathbf{k})$ space and eliminate $\mathbf{B}_{\mathbf{k}\omega}$
$\rightarrow$ $\mathcal{K}\cdot\mathbf{E} = 0$

Note that the cold plasma theory is obtained by $f(\mathbf{v})\rightarrow\delta(\mathbf{v})$

Example: $\mathbf{E}_0 = \mathbf{B}_0 = 0,\ f_0 = f_0(v^2)$ but consider the 3D problem:

Define:
$$F_{\alpha 0}(u) = \int f_{\alpha 0}\delta(u - \mathbf{k}\cdot\mathbf{v}/|k|)d^3v$$
$$E_{\mathbf{k}} = (\mathbf{k}\cdot\mathbf{E})/|k|$$
$$\mathbf{E}_\perp = (\mathbf{k}\times\mathbf{E})/|k|$$

The dispersion equation gets the form

$$\begin{bmatrix} K_\perp & 0 & 0 \\ 0 & K_\perp & 0 \\ 0 & 0 & K_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} E_{\perp 1} \\ E_{\perp 2} \\ E_{\mathbf{k}} \end{bmatrix} = 0$$

$$K_\perp = 1 - \frac{k^2c^2}{\omega^2} - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\int \frac{F_{\alpha 0}}{\omega - |k|u}du$$
$$K_{\mathbf{k}} = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega}\int_L \frac{F_{\alpha 0}/\partial u}{\omega/|k| - u}du$$

$$\text{electrostatic modes}: \quad K_{\mathbf{k}} = 0 \quad (\mathbf{E}_\perp = 0) \longleftarrow$$ Landau solution, ”longitudinal”

$$\text{electromagnetic modes}: \quad K_\perp = 0 \quad (\mathbf{E}_{\mathbf{k}} = 0) \longleftarrow$$ ”transversal”

$$\omega^2 = k^2c^2 + \sum_\alpha \omega_{p\alpha}^2 \int_{-\infty}^{\infty} \frac{\omega F_{\alpha 0}}{\omega - |k|u}du$$

For $\omega \gg kv_{the} \Rightarrow \omega^2 \approx k^2c^2 + \omega_{pe}^2$

the non-magnetized cold plasma EM wave

Uniformly magnetized plasma

$$\mathbf{B}_0 = B_0\mathbf{e}_z,\ \mathbf{E}_0 = 0,\ f_{\alpha 0} = f_{\alpha 0}(v_\perp^2, v_\parallel)$$
$$v_x = v_\perp\cos\phi,\ v_y = v_\perp\sin\phi,\ v_z = v_\parallel$$

Particle orbit:

$$\begin{aligned} v_x' &= v_\perp\cos(\phi - \omega_c\tau) &;\quad x' &= x - \frac{v_\perp}{\omega_c}\sin(\phi-\omega_c\tau) + \frac{v_\perp}{\omega_c}\sin\phi \\ v_y' &= v_\perp\sin(\phi - \omega_c\tau) &;\quad y' &= y + \frac{v_\perp}{\omega_c}\cos(\phi-\omega_c\tau) - \frac{v_\perp}{\omega_c}\cos\phi \\ v_z' &= v_\parallel &;\quad z' &= v_\parallel\tau + z\,. \end{aligned}$$

To integrate $f_{\alpha\mathbf{k}}$ we need a result from mathematical methods:

$$\exp\left(i\frac{k_\perp v_\perp}{\omega_c}\sin(\phi-\omega_c\tau)\right) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{k_\perp v_\perp}{\omega_c}\right)\exp[in(\phi-\omega_c\tau)]$$

Bessel functions of the first kind

Calculate a few pages $\Rightarrow$

$$\mathcal{K}(\omega, \mathbf{k}) = \left(1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\right)\mathcal{I} - \sum_\alpha \sum_{n=-\infty}^{\infty} \frac{2\pi\omega_{p\alpha}^2}{n_{\alpha 0}\omega^2}$$
$$\int_0^\infty \int_{-\infty}^{\infty} v_\perp dv_\perp dv_\parallel \left(k_\parallel \frac{\partial f_{\alpha 0}}{\partial v_\parallel} + \frac{n\omega_{c\alpha}}{v_\perp}\frac{\partial f_{\alpha 0}}{\partial v_\perp}\right) \frac{\mathcal{S}_{n\alpha}(v_\parallel, v_\perp)}{k_\parallel v_\parallel + n\omega_{c\alpha} - \omega}$$

$$\mathcal{S}_{n\alpha}(v_\parallel, v_\perp) = \begin{bmatrix} \frac{n^2\omega_{c\alpha}^2}{k_\perp^2} J_n^2 & \frac{inv_\perp\omega_{c\alpha}}{k_\perp} J_n J_n' & \frac{nv_\parallel\omega_{c\alpha}}{k_\perp} J_n^2 \\ -\frac{inv_\perp\omega_{c\alpha}}{k_\perp} J_n J_n' & v_\perp^2 J_n'^2 & -iv_\parallel v_\perp J_n J_n' \\ \frac{nv_\parallel\omega_{c\alpha}}{k_\perp} J_n^2 & iv_\parallel v_\perp J_n J_n' & v_\parallel^2 J_n^2 \end{bmatrix}$$

$J_n' = dJ_n/d(k_\perp v_\perp/\omega_{c\alpha})$

$\mathbf{B}_0$ makes the plasma anisotropic: Maxwellian $\rightarrow$ bi-Maxwellian

$$f_{\alpha 0} = \frac{m_\alpha}{2\pi k_B T_{\alpha\perp}} \sqrt{\frac{m_\alpha}{2\pi k_B T_{\alpha\parallel}}} \exp\left[-\frac{m_\alpha}{2k_B}\left(\frac{v_\perp^2}{T_{\alpha_\perp}} + \frac{v_\parallel^2}{T_{\alpha_\parallel}}\right)\right]$$

The distinction electrostatic – electromagnetic is no more exact
$\mathbf{E} \parallel \mathbf{k}$ is in many cases a good approximation
but also EM modes can have an $\mathbf{E}$ component parallel to $\mathbf{k}$

The Bessel functions introduce a harmonic mode structure $\omega \approx n\omega_{c\alpha}$

The resonance $\omega = \mathbf{k} \cdot \mathbf{v}$ in isotropic plasma is replaced by $\omega - n\omega_{c\alpha} = k_\parallel v_\parallel$
Only the velocity component $\parallel \mathbf{B}_0$ is involved in Landau damping
and only for $k_\parallel \neq 0$

Perpendicular propagation

$k_{\|} = 0$ i.e. $\theta = \pi/2$

$$\begin{bmatrix} K_{xx} & K_{xy} & 0 \\ K_{yx} & K_{yy} & 0 \\ 0 & 0 & K_{zz} \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0$$

$$K_{zz} = 1 - \frac{k^2c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_{\alpha}\sum_{n}\omega_{p\alpha}^2 \int_{-\infty}^{\infty} dv_{\|} \int_0^{\infty} \frac{J_n^2 f_{\alpha 0} v_{\perp}}{\omega - n\omega_{c\alpha}} dv_{\perp} = 0$$

This corresponds to the O-mode of cold plasma $\omega^2 \approx k^2c^2 + \omega_{pe}^2$

A new set of electrostatic cyclotron waves $\omega = n\omega_{c\alpha}\left\{1 + O\left[\frac{\omega_{p\alpha}^2}{k^2c^2}(kr_{L\alpha})^{2n}\right]\right\}$
(both ion and electron waves)

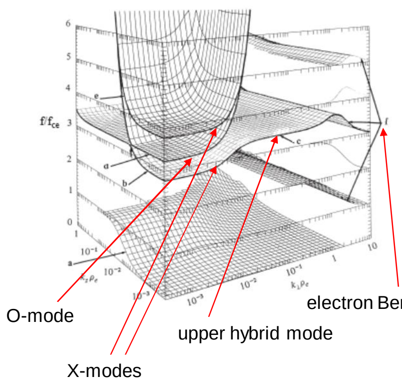
The remaining modes are the solutions of: $\begin{vmatrix} K_{xx} & K_{xy} \\ -K_{xy} & K_{yy} \end{vmatrix} = 0$

For these $\mathbf{E}\cdot\mathbf{k} \ll \mathbf{E}\times\mathbf{k}$ corresponding to the X-modes of cold plasma

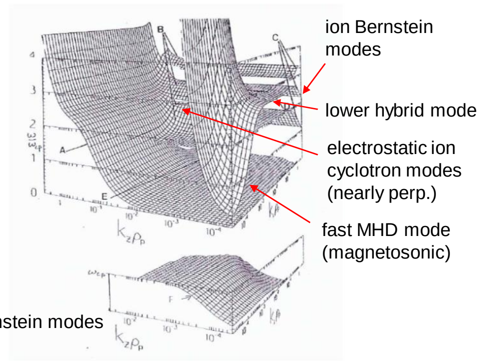
Another new set of modes with $\mathbf{E}\cdot\mathbf{k} \gg \mathbf{E}\times\mathbf{k}$
in the frequency bands $(n\omega_{c\alpha}, (n+1)\omega_{c\alpha})$ Bernstein modes

Perpendicular modes on dispersion surfaces

"electron modes"



"ion modes"



Parallel propagation $\theta = 0$

At $\omega \ll \omega_{ci}$ we find the Alfvén waves

$$\omega_r = \frac{k_\parallel v_A}{\sqrt{1 + v_A^2/c^2}}$$ ← with the "cold plasma correction"

Vlasov theory introduces damping also to the MHD modes

$$\omega_i = -\frac{\omega_{pi}^2}{|k_\parallel|}\frac{1}{1+c^2/v_A^2}\sqrt{\frac{\pi m_i}{8k_BT_i}}\exp\left(\frac{-B^2}{2\mu_0 n_e k_B T_i}\frac{\omega_{ci}^2}{\omega_r^2}\right)$$

$\omega \to \omega_{ci}$ from below: electromagnetic ion cyclotron wave (L mode)
damping by resonant ions

At higher frequencies we find the L and R modes (including whistlers)

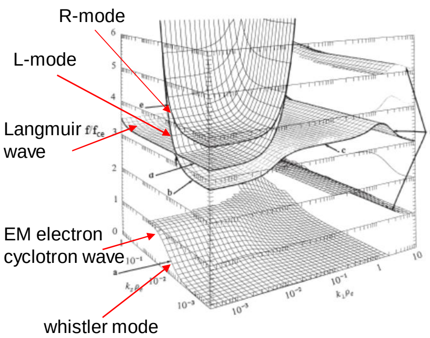
Near ω_{ce} whistler goes over to electromagnetic electron cyclotron wave

Note the difference of electromagnetic and electrostatic cyclotron waves:

Electromagnetic :	$\mathbf{k} \parallel \mathbf{B}_0$	$\omega \approx \omega_{c\alpha}$	no harmonic structure
Electrostatic :	$\mathbf{k} \perp \mathbf{B}_0$	$\omega \approx n\omega_{c\alpha}$	harmonic structure

Parallel modes on dispersion surfaces

"electron modes"



"ion modes"