Magnetohydrodynamics

Conservative form of MHD equations
Covection and diffusion
Frozen-in field lines
Magnetohydrostatic equilibrium
Magnetic field-aligned currents
Alfvén waves
Quasi-neutral hybrid approach

\[ \mathbf{J} \times \mathbf{B} = \nabla p \]

Basic MHD

\[ \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0 \]

\[ \rho_m \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) + \nabla p - \mathbf{J} \times \mathbf{B} = 0 \]

\[ \mathbf{E} + \mathbf{V} \times \mathbf{B} = -\frac{\mathbf{J} \times \mathbf{B}}{\mu_0} \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (\Rightarrow \nabla \cdot \mathbf{J} = 0) \]

Note that in collisionless plasmas \( \frac{1}{ne} \mathbf{J} \times \mathbf{B} \) and \( \frac{1}{ne} \nabla \cdot \mathbf{j} \) may need to be introduced to the ideal MHD's Ohm's law \( \mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \) before \( \mathbf{J}/\sigma \).
Some hydrodynamics

Conservation of mass density:
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0
\]

Navier-Stokes equation(s):
\[
\frac{\partial \mathbf{V}}{\partial t} = -\nabla P + \nu \nabla^2 \mathbf{V}
\]

Energy equation:
\[
\frac{d}{dt} \left( P \rho^\gamma \right) = 0,
\]
\(\gamma\) is the polytropic index, if \(\nu = 0\) these are called Euler equation(s)

In case \(\nu = 0\) we can write these in the conservation form \(\partial F/\partial t + \nabla \cdot \mathbf{G} = 0\)

\[
\begin{align*}
\frac{\partial P}{\partial t} & = -\nabla \cdot \mathbf{p} \quad \mathbf{p} = \rho \mathbf{V} \text{ is the momentum density} \\
\frac{\partial \mathbf{p}}{\partial t} & = -\nabla \left( \frac{\mathbf{p} \mathbf{p}}{\rho} + P \mathbf{T} \right) \quad \mathbf{p} \mathbf{p} \text{ is a tensor with components } p_i p_j \\
\frac{\partial u}{\partial t} & = -\nabla \left[ (u + P) \frac{\mathbf{p}}{\rho} \right] \quad u = \frac{P}{\gamma - 1} + \frac{p^2}{2 \rho} \quad \text{kinetic energy density}
\end{align*}
\]

(\(\rho, \mathbf{V}, P\)) primitive variables
(\(\rho, \mathbf{p}, u\)) conserved variables

MHD: add Ampère’s force

\[
\frac{\partial \mathbf{p}}{\partial t} = -\nabla \cdot \left( \frac{\mathbf{p} \mathbf{p}}{\rho} + P \mathbf{T} \right) + \mathbf{J} \times \mathbf{B}
\]

In MHD we neglect the displacement current, thus

\[
\begin{align*}
\mathbf{J} \times \mathbf{B} & = -\frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}) = -\nabla \left( \frac{B^2}{2 \mu_0} \right) + \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \mathbf{B}) \\
& \Rightarrow \frac{\partial \mathbf{p}}{\partial t} = -\nabla \cdot \left[ \frac{\mathbf{p} \mathbf{p}}{\rho} + \left( P + \frac{B^2}{2 \mu_0} \right) \mathbf{I} - \frac{1}{\mu_0} \mathbf{B} \mathbf{B} \right]
\end{align*}
\]

Now the energy density contains also the magnetic energy density

\[
u = \frac{P}{\gamma - 1} + \frac{p^2}{2 \rho} + \frac{B^2}{2 \mu_0}
\]

\[
\begin{align*}
\frac{\partial u}{\partial t} & = -\nabla \cdot \left[ (u + P) \frac{\mathbf{p}}{\rho} + \frac{1}{\mu_0 \rho} \mathbf{B} \times (\mathbf{p} \times \mathbf{B}) \right]
\end{align*}
\]
Ideal MHD: \( \mathbf{E} = -\nabla \times \mathbf{B} \)

\[ \Rightarrow \text{Faraday's law reduces to} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) \]

or using conserved variables:

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \frac{\mathbf{p}}{\rho} \times \mathbf{B} \right) \]

Thus we have found 8 equations in the conservation form

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{p} \]

\[ \frac{\partial \mathbf{p}}{\partial t} = -\nabla \cdot \left[ \rho \mathbf{u} + \left( P + \frac{\mathbf{B}^2}{2\mu_0} \right) I - \frac{1}{\mu_0} \mathbf{B} \mathbf{B} \right] \]

\[ \frac{\partial \mathbf{u}}{\partial t} = -\nabla \cdot \left[ \left( u + P \right) \mathbf{p} - \frac{3}{2\mu_0} \mathbf{B} \times (\mathbf{p} \times \mathbf{B}) \right] \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \frac{\mathbf{p}}{\rho} \times \mathbf{B} \right) \]

for 8 conserved variables \((\rho, \mathbf{p}, u, \mathbf{B})\)

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Convection (actually advection) and diffusion

Take curl of the MHD Ohm’s law \( \mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{J}/\sigma \) and apply Faraday’s law

\[ \Rightarrow \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{B} - \mathbf{J}/\sigma) \]

Thereafter use Ampère’s law and the divergence of \( \mathbf{B} \) to get the induction equation for the magnetic field

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \]

\[ \nabla \cdot \mathbf{B} = 0 \]

(Note that \( \sigma \) has been assumed constant)

Assume that plasma does not move \((\mathbf{V} = 0)\)

\[ \Rightarrow \text{diffusion equation:} \quad \frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B} \quad \text{diffusion coefficient:} \quad \eta = 1/\mu_0 \sigma \]

If the resistivity is finite, the magnetic field diffuses into the plasma to remove local magnetic inhomogeneities, e.g., curves in the field, etc.

Let \( L_B \) the characteristic scale of magnetic inhomogeneities. The solution is

\[ B = B_0 \exp(\pm t/\tau_d) \quad \text{where the characteristic diffusion time is} \quad \tau_d = \mu_0 \sigma L_B^2 \]
In case of $\sigma \to \infty$ the diffusion becomes very slow and the evolution of $B$ is completely determined by the plasma flow (field is frozen-in to the plasma)

$$\frac{dB}{dt} = \nabla \times (\mathbf{V} \times \mathbf{B})$$

convection equation

The measure of the relative strengths of convection and diffusion is the magnetic Reynolds number $R_m$

Let the characteristic spatial and temporal scales be $\nabla \rightarrow L_B^{-1}$ & $\partial/\partial t \rightarrow \tau^{-1}$ and the diffusion time $\tau_d = \mu_0 \sigma L_B^2$.

The order of magnitude estimates for the terms of the induction equation are

$$\frac{B}{\tau} = \frac{V_B + B}{L_B + \tau_d}$$

and the magnetic Reynolds number is given by $R_m = \mu_0 \sigma L_B V = L_B V/\eta$.

This is analogous to the Reynolds number in hydrodynamics: $R = LV/\nu$.

In fully ionized plasmas $R_m$ is often very large. E.g. in the solar wind at 1 AU it is $10^{16} - 10^{17}$. This means that during the 150 million km travel from the Sun the field diffuses about 1 km! Very ideal MHD: $E = -\mathbf{V} \times \mathbf{B}$.

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Example: Diffusion of 1-dimensional current sheet

$B(z, t) e_x$ in the frame co-moving with the plasma ($\mathbf{V} = 0$)

Initially: $B(z, 0) = \begin{cases} +B_0, & z > 0 \\ -B_0, & z < 0 \end{cases}$

$$\frac{\partial B}{\partial t} = \frac{\partial^2 B}{\partial z^2}$$

has the solution

$$B(z, t) = B_0 \operatorname{erf} \left( \frac{z}{\sqrt{4\eta t}} \right)$$

Total magnetic flux remains constant $= 0$, but the magnetic energy density $\int B^2/2\mu_0 dz$ decreases with time.

Now

$$\frac{\partial}{\partial t} \int \frac{B^2}{2\mu_0} dz = -\int \frac{\eta}{\sigma} dz$$

Ohmic heating, or Joule heating.
Example: Conductivity and diffusivity in the Sun

**Photosphere**
- partially ionized, collisions with neutrals cause resistivity:
\[ \sigma \approx 10 \, \Omega^{-1} m^{-1} \Rightarrow \eta \approx 10^5 \, m^2 s^{-1} \]

**Photospheric granules**
\[ L_B \approx 1000 \, \text{km}, \ V \approx 2 \, \text{km/s} \]
\[ R_m \approx 20000 \gg 1 \]

Observations of the evolution of magnetic structures suggest a factor of 200 larger diffusivity, i.e., smaller \( R_m \).

Explanation: Turbulence contributes to the diffusivity:
\[ \eta_r \approx 2 \cdot 10^7 \, m^2 s^{-1} \]

This is an empirical estimate, nobody knows how to calculate it!

**Corona**
- above 2000 km the atmosphere becomes fully ionized

Spitzer's formula for the effective electron collision time
\[ \tau_{ei}(s) = 0.266 \cdot 10^6 \frac{T^{3/2} (K)}{n_e (m^{-3})} \ln \Lambda \]

Rightarrow numerical estimate for the conductivity \( \sigma = \frac{n_e e^2 \tau_{ei}}{m_e} \)
\[ \sigma (\text{Sm}^{-1}) = 1.53 \cdot 10^{-2} \frac{T^{3/2} (K)}{\ln \Lambda} \]

Estimating \( \ln \Lambda = 20 \) the diffusivity becomes
\[ \eta (m^2 s^{-1}) = 10^3 \cdot T^{-3/2} (K) \]

For coronal temperature \( T = 10^6 \, \text{K} \)
\[ \eta = 1 \, m^2 / s \]

Home exercise: Estimate \( R_m \) in the solar wind close to the Earth
Frozen-in field lines

A concept introduced by Hannes Alfvén but later denounced by himself, because in the Maxwellian sense the field lines do not have physical identity. It is a very useful tool, when applied carefully.

Assume ideal MHD and consider two plasma elements joined at time $t$ by a magnetic field line and separated by $\Delta l$

In time $dt$ the elements move distances $u dt$ and $(u + \Delta u) dt$

In "Introduction to plasma physics" it was shown that the plasma elements are on a common field line also at time $t + dt$ $\Rightarrow$

In ideal MHD two plasma elements that are on a common field line remain on a common field line. In this sense it is safe to consider "moving" field lines.

Another way to express the freezing is to show that the magnetic flux through a closed loop defined by plasma elements is constant

Closed contour $C$ defined by plasma elements at time $t$

The same (perhaps deformed) contour $C'$ at time $t + \Delta t$

Arc element $dl$ moves in time $\Delta t$ the distance $V\Delta t$ and sweeps out an area $dl \times V\Delta t$

Consider a closed volume defined by surfaces $S$, $S'$ and the surface swept by $dl \times V\Delta t$ when $dl$ is integrated along the closed contour $C$.

As $V \cdot B = 0$, the magnetic flux through the closed surface must vanish at time $t + \Delta t$, i.e,

$$\int_S B(t + \Delta t) \cdot dS + \int_{S'} B(t + \Delta t) \cdot dS' + \int_C B(t + \Delta t) \cdot dl \times V \Delta t = 0$$

Calculate $d\Phi/dt$ when the contour $C$ moves with the fluid:
The integrand vanishes, if \( \frac{d\Phi}{dt} = \lim_{\Delta t \to 0} \frac{\Phi_C(t + \Delta t) - \Phi_C(t)}{\Delta t} \)

\[ = \frac{\int [\mathbf{B}(t + \Delta t) - \mathbf{B}(t)] \cdot dS}{\Delta t} \]

\[ = \int \frac{\partial \mathbf{B}}{\partial t} \cdot dS - \oint_{C} (\mathbf{V} \times \mathbf{B}) \cdot d\mathbf{l} \]

\[ = \int \left[ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{B}) \right] \cdot dS \]

\[ = - \int \nabla \times (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot dS. \]

The integrand vanishes, if \( \mathbf{E} + \mathbf{V} \times \mathbf{B} = -\nabla \Psi \)

Thus in ideal MHD the flux through a closed contour defined by plasma elements is constant, i.e., plasma and the field move together.

The critical assumption was the ideal MHD Ohm’s law. This requires that the ExB drift is faster than magnetic drifts (i.e., large scale convection dominates). As the magnetic drifts lead to the separation of electron and ion motions \( (\mathbf{J}) \), the first correction to Ohm’s law in collisionless plasma is the Hall term

\[ \mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{1}{ne} \mathbf{J} \times \mathbf{B} \]

so-called Hall MHD

It is a straightforward exercise to show that in Hall MHD the magnetic field is frozen-in to the electron motion and \( \mathbf{E} = -\mathbf{V}_e \times \mathbf{B} \)

When two ideal MHD plasmas with different magnetic field orientations flow against each other, magnetic reconnection can take place. Magnetic reconnection can break the frozen-in condition in an explosive way and lead to rapid particle acceleration.

Example of reconnection: a solar flare
Magnetohydrostatic equilibrium

Assume scalar pressure \( \nabla \cdot P = 0 \) and \( d/dt = 0 \)

\[ \Rightarrow \quad J \times B = \nabla P \quad \Rightarrow \quad \begin{cases} B \cdot \nabla P = 0 \\ J \cdot \nabla P = 0 \end{cases} \]

i.e., \( B \) and \( J \) are vector fields on constant pressure surfaces

Write the magnetic force as:

\[ J \times B = -\frac{1}{\mu_0} B \times (\nabla \times B) \]

\[ = -\nabla \left( \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} \nabla \cdot (B B) \] magnetic stress and torsion

Magnetohydrostatic equilibrium after elimination of the current

\[ \nabla \cdot P = -\frac{1}{\mu_0} B \times (\nabla \times B) \]

Assume scalar pressure and negligible \( \nabla \cdot (BB) \)

\[ \nabla \left( P + \frac{B^2}{2\mu_0} \right) = 0 \]

Plasma beta:

\[ \beta = \frac{2\mu_0 P}{B^2} \]

The current perpendicular to \( B \):

\[ J_\perp = \frac{B \times \nabla P}{B^2} \]

**Diamagnetic current**: this is the way how \( B \) reacts on the presence of plasma in order to reach magnetohydrostatic equilibrium.

This macroscopic current is a sum of drift currents and of the magnetization current due to inhomogeneous distribution of elementary magnetic moments (particles on Larmor motion)

\[ J_M = \nabla \times M \]

In magnetized plasmas pressure and temperature can be anisotropic \( (\beta_\perp \neq \beta_\parallel) \)

Define parallel and perpendicular pressures as \( P_\perp = nW_\perp \) and \( P_\parallel = 2nW_\parallel \)

Using these we can derive the macroscopic currents from \( J_C = \frac{P_\perp}{B} (\nabla \times b)_\perp \) where \( b = B/B \)

\[ J_G = P_\parallel \nabla \frac{1}{B} \times b \]
The magnetization is \( \mathbf{M} = n \mathbf{\mu} = -n \mathbf{W}_\perp / B \mathbf{b} \)

\[ \Rightarrow \mathbf{J}_M = \nabla \times \mathbf{M} = -\nabla \left( \frac{P_\perp}{B} \mathbf{b} \right) \]

Summing up \( \mathbf{J}_G, \mathbf{J}_C \) and \( \mathbf{J}_M \) we get the total current

\[ \mathbf{J} = \frac{\mathbf{B} \times \nabla P}{B^2} + \frac{P_\parallel - P_\perp}{B} \nabla \times \mathbf{b} \]

Magnetohydrostatic equilibrium in anisotropic plasma is described by

\[ \mathbf{J} \times \mathbf{B} = \nabla \cdot (P_\parallel + (P_\parallel - P_\perp) \mathbf{b} \cdot \nabla \mathbf{b}) = (\nabla \cdot P)_\perp \\
(\nabla \cdot P)_\parallel = 0 \]

In time dependent problems the polarization current must be added

\[ \mathbf{J}_P = \frac{\rho_m}{B^2} \frac{d \mathbf{E}}{dt} \]

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**Force-free fields a.k.a. magnetic field-aligned currents**

In case of \( \beta \ll 1 \) the pressure gradient is negligible and the equilibrium is \( \mathbf{J} \times \mathbf{B} = 0 \)
i.e., no Ampère’s force

\[ \mathbf{J} \times \mathbf{B} = 0 \]  is a non-linear equation \( (\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \)

If \( B_1 \) and \( B_2 \) are two solutions, \( B_1 + B_2 \) does not need to be another solution

Now \( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mathbf{\alpha}(\mathbf{r}) \mathbf{B} \Rightarrow \mathbf{B} \cdot \nabla \alpha = 0 \)  \( \alpha \) is constant along \( B \)

If \( \alpha \) is constant in all directions, the equation becomes linear, the **Helmholtz equation**

\[ \nabla^2 \mathbf{B} + \alpha^2 \mathbf{B} = 0 \]

Note that potential fields \( \mathbf{B} = \nabla \psi \) are trivially force-free
Force-free fields as flux-ropes in space

Loops in the Solar corona

Coronal Mass Ejection (CME)

Interplanetary Coronal Mass Ejection (ICME)

Example: Linear force-free model of a coronal arcade

Construct a model that
• looks like an arc in the $x$-$z$-plane
• extends uniformly in the $y$-direction

The form of the Helmholtz equation

$$\nabla^2 \mathbf{B} + \alpha^2 \mathbf{B} = 0$$

suggests a trial of the form:

$$\begin{cases} B_x &= B_{x,0} \sin(kx)e^{-lz} \\ B_y &= B_{y,0} \sin(kx)e^{-lz} \\ B_z &= B_0 \cos(kx)e^{-lz} \end{cases}$$

with the condition

$$\alpha^2 < k^2$$

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \implies \begin{cases} l B_{x,0} &= \alpha B_{x,0} \\ -l B_{x,0} + k B_0 &= \alpha B_{x,0} \\ k B_{x,0} &= \alpha B_0 \end{cases} \implies \begin{cases} B_x &= (l/k) B_0 \sin(kx)e^{-lz} \\ B_y &= (\alpha/k) B_0 \sin(kx)e^{-lz} \\ B_z &= B_0 \cos(kx)e^{-lz} \end{cases}$$

$$l^2 = k^2 - \alpha^2$$
Projection of field lines in the $xy$-plane are straight lines parallel to each other.

The solution looks like:

Arcs in the $xz$-plane

Case of so weak current that we can neglect it ($\alpha = 0$):
Potential field (no distortion due to the current)

Solve the two-dimensional Laplace equation

$$ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 $$

Look for a separable solution $\Psi = X(x)Z(z)$

Magnetic field lines are now arcs

$$ B_x = \frac{\partial \Psi}{\partial x} = B_0 \cos kx \exp(-kz) $$
$$ B_z = \frac{\partial \Psi}{\partial z} = -B_0 \sin kx \exp(-kz) $$

without the shear in the $y$-direction

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**Grad – Shafranov equation**

Often a 3D structure is essentially 2D (at least locally)

Consider translational symmetry

$B$ uniform in $z$ direction

$$ \nabla \cdot B = 0 \Rightarrow B = \left( \frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, B_z \right) $$

$$ A = A(x,y) e_z $$

Assume balance btw. magnetic and pressure forces

$$ \frac{1}{\mu_0} (\nabla \times B) \times B - \nabla P = 0 $$

The $z$ component reduces to

$$ \frac{\partial B_z}{\partial x} \frac{\partial A}{\partial y} - \frac{\partial B_z}{\partial y} \frac{\partial A}{\partial x} = 0 \Rightarrow \nabla \perp B_z \parallel \nabla \perp A $$

$$ B_z(x,y) = B_z(A(x,y)) $$
$B_z(x, y) = B_z(A(x, y)) \Rightarrow \frac{1}{\mu_0} (B_zB'_z + \nabla^2_A A) \frac{\partial A}{\partial x} + \frac{\partial P}{\partial x} = 0$

Now also

$\nabla_A P \parallel \nabla_A A$

$\Rightarrow P(x, y) = P(A(x, y)) \Rightarrow \frac{1}{\mu_0} (\nabla^2_A A + B_zB'_z) + P' = 0$

Total pressure

$P_1 = \frac{B^2}{2\mu_0} + P$

$\Rightarrow$ the Grad-Shafranov equation

$\frac{1}{\mu_0} \nabla^2_A A + \frac{d}{dA} P_1 = 0$

While non-linear, it is a scalar eq.

The GS equation is not force-free.

For a force-free configuration we can set $P = 0 \Rightarrow$

$\frac{1}{\mu_0} \nabla^2_A A + \frac{d}{dA} \left( \frac{B^2}{2\mu_0} \right) = 0$

Examples of using the Grad-Shafranov method in studies of the structure of ICMEs, see, Isavnin et al., Solar Phys., 273, 205-219, 2011

DOI: 10.1007/s11207-011-9845-z

Black arrows: Magnetic field measured by a spacecraft during the passage of an ICME

Contours: Reconstruction of the ICME assuming that GS equation is valid
Some general properties of force-free fields

1. A field with finite magnetic energy cannot be force-free everywhere.

\[
W = \int \frac{B^2}{2\mu_0} dV = \frac{1}{\mu_0} \int r \cdot J \times B \, dV
\]

Proof: As \( B \) vanishes faster than \( r^{-2} \) at large \( r \), we can write

Proof: Exercise

2. If \( \mathbf{J} \times \mathbf{B} = 0 \) in a finite volume \( V \) and on its boundary \( S \), then \( \mathbf{B} = 0 \) everywhere.

Proof: Exercise

Thus if there is FAC in a finite volume, it must be anchored to perpendicular currents at the boundary of the volume (e.g. magnetosphere-ionosphere coupling through FACs)

\[
\nabla \cdot \mathbf{J} = \nabla_{\|} \cdot \mathbf{J} + \nabla_{\perp} \cdot \mathbf{J} = 0
\]

3. An axisymmetric, force-free, poloidal magnetic field must be current-free.

Magnetospheric and ionospheric currents are coupled through field-aligned currents (FAC)

Solar energy

\[
\text{Magnetosphere (transports and stores energy)}
\]

FAC

\[
\text{Ionosphere (dissipate energy)}
\]

"The ionosphere and magnetosphere are coupled in so many different ways that nearly every magnetospheric process bears on the ionosphere in some way and every ionospheric process on the magnetosphere"

Wolf, 1974
Recall: Poynting’s theorem

EM energy is dissipated via currents $\mathbf{J}$ in a medium.

The rate of work is defined $\mathbf{F} \cdot \mathbf{v} = q \mathbf{E} \cdot \mathbf{v} = \mathbf{J} \cdot \mathbf{E}$

$\mathbf{J} \perp \mathbf{E}$ no energy dissipation!

$$ - \int_V \mathbf{J} \cdot \mathbf{E} \, d^3r = \int_{\partial V} \mathbf{S} \cdot d\mathbf{a} + \int_V \frac{\partial}{\partial t} (w_E + w_M) \, d^3r $$

Poynting vector $(W/m^2)$ giving the flux of EM energy

$\nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}$

$\mathbf{E} \cdot \mathbf{J} > 0$: magnetic energy converted into kinetic energy (e.g. reconnection)

$\mathbf{J} \cdot \mathbf{E} < 0$: kinetic energy converted into magnetic energy (dynamo)

How are magnetospheric currents produced?

Here ($\mathbf{J} \cdot \mathbf{E} < 0$), i.e., the magnetopause opened by reconnection acts as a generator (Poynting flux into the magnetosphere)

Dayside reconnection is a load ($\mathbf{J} \cdot \mathbf{E} > 0$) and does not create current

Solar wind pressure maintains $\mathbf{J}_F$ and the magnetopause current
Currents in the magnetosphere

In large scale (MHD) magnetospheric currents are source-free
\( \nabla \cdot \mathbf{J} = 0 \) and obey Ampère’s law
\( \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \)

Thus, if we know \( \mathbf{B} \) everywhere, we can calculate the currents (in MHD \( \mathbf{J} \) is a secondary quantity!).
However, it is not possible to determine \( \mathbf{B} \) everywhere, as variable plasma currents produce variable \( \mathbf{B} \)

Recall the electric current in (anisotropic) magnetohydrostatic equilibrium:
\[
\mathbf{J} = \frac{\mathbf{B} \times \nabla P}{B^2} + \frac{P_{\parallel} - P_{\perp}}{B} (\nabla \times \mathbf{b})_\perp
\]

In addition, a time-dependent electric field drives polarization current:
\[
J_{pol} = \frac{\rho_{pol}}{B^2} \frac{\partial \mathbf{E}}{\partial t}
\]

These are the main macroscopic currents (\( \perp \mathbf{B} \)) in the magnetosphere

A simple MHD generator

Plasma flows across the magnetic field
\[
\mathbf{F} = q (\mathbf{u} \times \mathbf{B}) \rightarrow e^- \text{ down, } i^+ \text{ up}
\]

Let the electrodes be coupled through a load \( \rightarrow \) current loop, where the current \( \mathbf{J} \) through the plasma points upward

Now Ampère’s force \( \mathbf{J} \times \mathbf{B} \) decelerates the plasma flow, i.e., the energy to the newly created \( \mathbf{J} \) (and thus \( \mathbf{B} \)) in the circuit is taken from the kinetic energy.

Kinetic energy \( \rightarrow \) EM energy: Dynamo!

The setting can be reversed by driving a current through the plasma: \( \mathbf{J} \times \mathbf{B} \) accelerates the plasma
\( \rightarrow \) plasma motor
**Boundary layer generator**

One scenario how an MHD generator may drive field-aligned current from the LLBL:

Plasma penetrates to the LLBL through the dayside magnetopause (e.g. by reconnection.

The charge separation ("electrodes" of the previous example) is consistent with \( \mathbf{E} = - \mathbf{V} \times \mathbf{B} \).

The FAC ion the inside part closes via ionospheric currents (the so-called Region 1 current system).

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**FAC from the magnetosphere**

We start from the (MHD) current continuity equation

\[ \nabla \cdot \mathbf{J} = \nabla_\perp \cdot \mathbf{J}_\perp + \nabla_\parallel \cdot \mathbf{J}_\parallel = 0 \quad \text{where} \quad \nabla_\parallel \cdot \mathbf{J}_\parallel = \mathbf{B} \cdot \nabla (\mathbf{J}_\parallel / \mathbf{B}) \]

The divergence of the static total perpendicular current is

\[ \nabla_\perp \cdot \mathbf{J} = \nabla_\perp \cdot (\mathbf{J}_C + \mathbf{J}_G) = \nabla_\perp \cdot \left( \frac{\mathbf{p}_\perp - \mathbf{p}_\parallel}{\mathbf{B}} \right) \nabla \times \mathbf{b} - \nabla_\perp \cdot \mathbf{J}_\parallel \nabla \times \frac{\mathbf{b}}{\mathbf{B}} \]

In the isotropic case this reduces to

\[ \nabla_\perp \cdot \mathbf{J} = \nabla_\perp \cdot \left( \frac{- \nabla p \times \mathbf{b}}{\mathbf{B}} \right) \]

The polarization current can have also a divergence

\[ \nabla_\perp \cdot \mathbf{J}_{\text{pol}} = \nabla_\perp \cdot \left( \frac{\mathbf{p}_\text{pol}}{\mathbf{B}^2} \nabla \times \mathbf{E} \right) = \frac{\mathbf{p}_\text{pol}}{\mathbf{B}} \nabla \times \mathbf{V} = \frac{\mathbf{p}_\text{pol}}{\mathbf{B}} \frac{\partial \Omega}{\partial t} = \mathbf{b} \cdot (\nabla \times \mathbf{V}) \]

where \( \Omega \) is the vorticity

Integrating the continuity equation along a field line we get:

\[ \mathbf{J}_\parallel = - \frac{\mathbf{b}}{\mathbf{B}} \int \frac{\mathbf{J}_\perp}{\mathbf{B}} d\ell = - \frac{\mathbf{b}}{\mathbf{B}} \int \left[ \nabla_\perp \cdot \left( \frac{\nabla \times \frac{\mathbf{b}}{\mathbf{B}}}{\mathbf{B}} \right) - \frac{\mathbf{p}_\text{pol}}{\mathbf{B}} \frac{\partial \Omega}{\partial t} \right] d\ell \]

Sources of FACs: pressure gradients & time-dependent \( \Omega \).
FAC from the ionosphere

Write the ionospheric current as a sum of Pedersen and Hall currents and integrate the continuity equation along a field line:

\[ J_B = - \int \nabla_\perp \cdot \left( \sigma_P E - \sigma_H \frac{E \times B}{B} \right) \, dz \]

Approximate the RHS integral as

\[ J_B \approx - \nabla_\perp \cdot \left( \Sigma_P E - \Sigma_H \frac{E \times B}{B} \right) = - \nabla_\perp \cdot (\Sigma_P E) + \frac{E \times B}{B} \cdot \nabla \Sigma_H \]

where \( \Sigma_P = h \sigma_P \) and \( \Sigma_H = h \sigma_H \) are height-integrated Pedersen and Hall conductivities and \( h \) the thickness of the conductive ionosphere (~ the E-layer)

The sources and sinks of FACs in the ionosphere are divergences of the Pedersen current and gradients of the Hall conductivity

Recall:

\[ \nabla \cdot \mathbf{J} = \nabla_\perp \cdot \mathbf{J}_B + \nabla_\parallel \cdot \mathbf{J}_L = 0 \]
Current closure

The closure of the current systems is a non-trivial problem. Let’s make a simple exercise based on continuity equation.

Let the system be isotropic. The magnetospheric current is

\[ \mathbf{J}_{MS} = \frac{\mathbf{B} \times \nabla p}{B^2} - \frac{\rho_m}{B^2} \frac{d}{dt} \left( \frac{\Omega}{B} \right) \mathbf{B} \]  

(the last term contains inertial currents)

\[ \Rightarrow \nabla \cdot \mathbf{J}_{MS} = \nabla \cdot \left( \frac{\rho_m}{B^2} \left( \frac{\Omega}{B} \right) \mathbf{B} - \frac{\rho_m}{B^2} \frac{d}{dt} \left( \frac{\Omega}{B} \right) \mathbf{B} \right) \]  

we neglect this

Integrate this along a field line from one ionospheric end to the other

\[ \int_{z_1}^{z_2} \frac{1}{B} \left\{ -2 \mathbf{b} \cdot [\nabla p \times \nabla (1/B)] + \rho_m \frac{d}{dt} \left( \frac{\Omega}{B} \right) \right\} dz \]

Assuming the magnetic field north-south symmetric, this reduces to

\[ \frac{J_{H}}{B_I} \approx \frac{1}{2} \int \left\{ \mathbf{b} \cdot [\nabla p \times \nabla (1/B^2)] - \frac{\rho_m}{B^2} \frac{d \Omega}{dt} \right\} dz \]

This must be the same as the FAC from the ionosphere and we get an equation for ionosphere-magnetosphere coupling

\[ \nabla \cdot (\nabla p \varphi) + \frac{B_I \times \nabla \varphi}{B_I} \cdot \nabla \Sigma_H = \frac{B_I}{2} \int \left\{ \mathbf{b} \cdot [\nabla p \times \nabla (1/B^2)] - \frac{\rho_m}{B^2} \frac{d \Omega}{dt} \right\} dz \]

One more view on M-I coupling

![Diagram showing M-I coupling](image-url)
Magnetic helicity

\[ H = \int \mathbf{A} \cdot \mathbf{B} \, dV \]  
\( \mathbf{A} \) is the vector potential

Gauge transformation:
\[ H \rightarrow H' = H + \int \mathbf{B} \cdot \nabla \chi \, dV \]
\[ A \rightarrow A' = A + \nabla \chi \]
Helicity is gauge-independent only if the field extends over all space and vanishes sufficiently rapidly

For finite magnetic field configurations helicity is well defined if and only if
\[ \mathbf{B} \cdot \mathbf{n} = 0 \] on the boundary

Helicity is a conserved quantity if
\[ \bullet \] the field is confined within a closed surface \( S \) on which \( \mathbf{B} \cdot \mathbf{n} = 0 \)
\[ \bullet \] the field is in a perfectly conducting medium for which \( \mathbf{B} \cdot \mathbf{V} = 0 \) on \( S \)

To prove this, note that from
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) \]
we get
\[ \frac{\partial \mathbf{A}}{\partial t} = \mathbf{V} \times \mathbf{B} \] to within a gauge transformation

Calculate a little
\[
\frac{dH}{dt} = \int \left( \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) \, dV \\
= \int \left[ \frac{\partial \mathbf{A}}{\partial t} \cdot (\nabla \times \mathbf{A}) + \mathbf{A} \cdot \nabla \times \left( \frac{\partial \mathbf{A}}{\partial t} \right) \right] \, dV \\
= \int \nabla \cdot \left( \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{A} \right) \, dV \\
= \int \mathbf{n} \cdot \left( \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{A} \right) \, dS \\
= 0, \\
\text{and both } \mathbf{B} \text{ and } \mathbf{V} \text{ are } \mathbf{n} \text{ on } S \]

\[ \text{thus } \frac{\partial \mathbf{A}}{\partial t} \parallel \mathbf{n} \text{ on } S \]
Example: Helicity of two flux tubes linked together

\[ H = H_1 + H_2 \]

For thin flux tubes \( \mathbf{B} = \nabla \times \mathbf{A} \)

is approximately normal to the cross section \( S \)

(note: here \( S \) is not the boundary of \( V \))

Helicity of tube 1:

\[ H_1 = \int A \cdot B \, dV = \int ds \cdot A \left( \int dS \mathbf{n} \cdot \nabla \times \mathbf{A} \right) \]

Thus \( H_1 = \Phi_1 \Phi_2 \)

Similarly \( H_2 = \Phi_1 \Phi_2 \Rightarrow H = 2\Phi_1 \Phi_2 \)

If the tubes are wound \( N \) times around each other

\( H = \pm 2N\Phi_1 \Phi_2 \)

Complex flux-tubes on the Sun

Helicity is a measure of structural complexity of the magnetic field
Woltjer’s theorem

For a perfectly conducting plasma in a closed volume $V_0$ the integral

$$\int_{V_0} \mathbf{A} \cdot \mathbf{B} \, dV = H_0$$

is invariant and the minimum energy state is a linear force-free field.

Proof: Invariance was already shown.

To find the minimum energy state consider

$$W = \int_{V_0} \frac{B^2}{2\mu_0} \, dV$$

Small perturbations: $\mathbf{A} + \delta \mathbf{A} \quad \mathbf{B} + \delta \mathbf{B}$

$$\delta \mathbf{A} = 0 \quad \text{on} \quad \partial \mathbf{S} \quad \text{and} \quad \delta \mathbf{B} = \nabla \times \delta \mathbf{A}$$

$$2\mu_0 \delta W = \int_{V_0} \left[ 2\mathbf{B} \cdot \delta \mathbf{B} - \alpha_0 (\delta \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \delta \mathbf{B}) \right] \, dV$$

$$= \int_{V_0} \nabla \cdot (\nabla \times \delta \mathbf{A} + 2\alpha_0 \mathbf{A} \times \delta \mathbf{A}) \, dV$$

$$+ 2 \int_{V_0} (\nabla \times \delta \mathbf{B} - \alpha_0 \mathbf{B}) \cdot \delta \mathbf{A} \, dV.$$

Thus $\delta W = 0$ if and only if $\nabla \times \mathbf{B} = \alpha_0 \mathbf{B}$

Magnetohydrodynamic waves

• Dispersion equation for MHD waves
• Alfvén wave modes

MHD is a fluid theory and there are similar wave modes as in ordinary fluid theory (hydrodynamics). In hydrodynamics the restoring forces for perturbations are the pressure gradient and gravity. Also in MHD the pressure force leads to acoustic fluctuations, whereas Ampère’s force ($\mathbf{J} \times \mathbf{B}$) leads to an entirely new class of wave modes, called Alfvén (or MHD) waves.

As the displacement current $c^{-2} \partial \mathbf{E}/\partial t$ is neglected in MHD, there are no electromagnetic waves of classical electrodynamics. Of course EM waves can propagate through MHD plasma (e.g. light, radio waves, etc.) and even interact with the plasma particles, but that is beyond the MHD approximation.
Dispersion equation for ideal MHD waves

\[
\begin{align*}
\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) &= 0 \\
\rho_m \frac{\partial \mathbf{V}}{\partial t} + \rho_m (\mathbf{V} \cdot \nabla) \mathbf{V} &= -\nabla P + \mathbf{J} \times \mathbf{B} \\
\nabla P &= \frac{\mu_0}{\rho_m} \mathbf{J} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\mathbf{E} + \mathbf{V} \times \mathbf{B} &= 0
\end{align*}
\]

\[
\Rightarrow \quad \frac{\partial \mathbf{V}}{\partial t} + \rho_m (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla \frac{\mu_0}{\rho_m} P + (\nabla \times \mathbf{B}) \times \mathbf{B} / \mu_0
\]

\[
\nabla \times (\mathbf{V} \times \mathbf{B}) = \frac{\partial \mathbf{B}}{\partial t}.
\]

We are left with 7 scalar equations for 7 unknowns ($\rho_m$, $\mathbf{V}$, $\mathbf{B}$)

Consider small perturbations:

\[
\begin{align*}
\mathbf{B}(r, t) &= \mathbf{B}_0 + \mathbf{B}_1(r, t) \\
\rho_m(r, t) &= \rho_{m0} + \rho_{m1}(r, t) \\
\mathbf{V}(r, t) &= \mathbf{V}_1(r, t).
\end{align*}
\]

and linearize

Consider the simple case when $\sigma = \omega$, $\mu = 1$ and the imposed constant magnetic field $\mathbf{H}_0$ is homogeneous and parallel to the $x$-axis. In order to study a plane wave we assume that all variables depend upon the time $t$ and $z$ only. If the velocity $x$ is parallel to the $x$-axis, the current $i$ is parallel to the $y$-axis and produces a variable magnetic field $\mathbf{B}'$ in the $x$-direction. By elementary calculation we obtain

\[
\frac{\partial \mathbf{B}'}{\partial t} = \frac{\partial \mathbf{B}'}{\partial t} = \mu_0 \frac{\partial \mathbf{B}'}{\partial t}.
\]

which means a wave in the direction of the $x$-axis with the velocity

\[
v = H_0 / \sqrt{\mu_0}.
\]

Waves of this sort may be of importance in solar physics. As the sun has a general magnetic field, and a solar matter is a good conductor, the conditions for the existence of electromagnetic-hydrodynamic waves are satisfied. If in a region of the sun we have $H_0 = 10$ gauss and $P = 0.005$ mm. Hg. m., the velocity of the waves amounts to

\[
v \sim 10^4 \text{ cm. sec.}^{-1}.
\]

This is about the velocity with which the sunspot zone moves towards the equator during the sunspot cycle. The above values of $H_0$ and $\sigma$ refer to a distance of about one million cm. below the solar surface where the original cause of the sunspots may be found. Thus it is possible that the sunspots are associated with a magnetic and mechanical disturbance preceding an electromagnetic-hydrodynamic wave.

The matter is further discussed in a paper which will appear in Arkiv för matematik, Astronomi och Fysik.

Kgl. Tekniska Högskolan,
Stockholm,
Aug. 24.
\[ \frac{\partial P_{m1}}{\partial t} + \rho_{m0}(\nabla \cdot \mathbf{V}_1) = 0 \quad (*) \]
\[ \rho_{m0} \frac{\partial \mathbf{V}_1}{\partial t} + c_s^2 \nabla \rho_{m1} + B_0 \times (\nabla \times \mathbf{B}_1)/\rho_0 = 0 \quad (**) \]
\[ \frac{\partial \mathbf{B}_1}{\partial t} - \nabla \times (\mathbf{V}_1 \times \mathbf{B}_0) = 0 \quad (***) \]

Find an equation for \( \mathbf{V}_1 \). Start by taking the time derivative of (**) \[ \frac{\partial^2 \mathbf{V}_1}{\partial t^2} + c_s^2 \nabla \left( \frac{\partial \rho_{m1}}{\partial t} \right) + \frac{B_0}{\rho_0} \times \left( \nabla \times \frac{\partial \mathbf{B}_1}{\partial t} \right) = 0 \]

Insert (*) and (***) and introduce the Alfvén velocity as a vector \( \mathbf{V}_A = \frac{B_0}{\sqrt{\rho_0 \mu_0}} \)

Look for plane wave solutions \( \mathbf{V}_1(r, t) = \mathbf{V}_1 \exp\{i(k \cdot r - \omega t)\} \)

Now and the dispersion equation reduces to

\[ -\omega^2 \mathbf{V}_1 + (c_s^2 + v_A^2)(\mathbf{k} \cdot \mathbf{V}_1)\mathbf{k} = 0 \]

Using \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \) a few times we have the dispersion equation for the waves in ideal MHD

\[ \omega/k = \sqrt{v_A^2 + c_s^2} \]

**Alfvén wave modes**

**Propagation perpendicular to the magnetic field:** \( \mathbf{k} \perp \mathbf{B}_0 \)

Now \( \mathbf{k} \cdot \mathbf{V}_A = 0 \) and the dispersion equation reduces to

\[ -\omega^2 \mathbf{V}_1 + (c_s^2 + v_A^2)(\mathbf{k} \cdot \mathbf{V}_1)\mathbf{k} = 0 \]

\[ \Rightarrow \quad \mathbf{V}_1 = (c_s^2 + v_A^2)(\mathbf{k} \cdot \mathbf{V}_1)/\omega^2 \quad \text{clearly} \quad \mathbf{k} \parallel \mathbf{V}_1 \]

And we have found the **magnetosonic wave** \( \omega/k = \sqrt{v_A^2 + c_s^2} \)

Making a plane wave assumption also for \( \mathbf{B}_1 \) (very reasonable, why?)

\[ \omega \mathbf{B}_1 + \mathbf{k} \times (\mathbf{V}_1 \times \mathbf{B}_0) = 0 \quad \Rightarrow \quad \mathbf{B}_1 = \frac{\mathbf{V}_1}{\omega/k} \mathbf{B}_0 \quad \text{(i.e.,} \mathbf{B}_1 \parallel \mathbf{B}_0) \]

The wave electric field follows from the frozen-in condition

\[ \mathbf{E} = -\mathbf{V}_1 \times \mathbf{B}_0 \]

This mode has many names in the literature:

- Compressional Alfvén wave
- Fast Alfvén wave
- Fast MHD wave
Propagation parallel to the magnetic field: \( k \parallel B_0 \)

Now the dispersion equation reduces to

\[
(k^2 v_A^2 - \omega^2)V_1 + \left( \frac{v_A^2}{v_A^2} - 1 \right) k^2 (V_1 \cdot v_A) v_A = 0
\]

Two different solutions (modes)

1) \( V_1 \parallel B_0 \parallel k \Rightarrow \omega/k = v_s \)

   - the sound wave

2) \( V_1 \perp B_0 \parallel k \Rightarrow V_1 \cdot v_A = 0 \)

   \[ \Rightarrow \omega/k = v_A \]

This mode is called

- Alfvén wave or
- shear Alfvén wave

Propagation at an arbitrary angle

- \( k = k(e_x \sin \theta + e_z \cos \theta) \)
- \( v_A = v_A e_z \)
- \( V_1 = V_{1x} e_x + V_{1y} e_y + V_{1z} e_z \)
- \( k \cdot v_A = k v_A \cos \theta \)
- \( k \cdot V_1 = k (V_{1x} \sin \theta + V_{1z} \cos \theta) \)
- \( v_A \cdot V_1 = v_A V_{1x} \).

Dispersion equation \( \Rightarrow \)

\[
V_{1x} (-\omega^2 + k^2 v_A^2 + k^2 v_x^2 \sin^2 \theta) + V_{1x} (k^2 v_A^2 \sin \theta \cos \theta) = 0
\]

\[
V_{1y} (-\omega^2 + k^2 v_A^2 \cos^2 \theta) = 0
\]

\[
V_{1x} (k^2 v_A^2 \sin \theta \cos \theta) + V_{1x} (-\omega^2 + k^2 v_x^2 \cos^2 \theta) = 0
\]

Coeff. of \( V_{1y} \Rightarrow \omega/k = v_A \cos \theta \) shear Alfvén wave

From the determinant of the remaining equations:

\[
\left( \frac{\omega}{k} \right)^2 = \frac{1}{2} (v^2_x + v_A^2) \pm \frac{1}{2} \sqrt{(v^2_x + v_A^2)^2 - 4 v^2_x v_A^2 \cos^2 \theta}^{1/2}
\]

Fast (+) and slow (–) Alfvén/MHD waves
Beyond MHD: Quasi-neutral hybrid approach

MHD is not an appropriate description if the physical scale sizes of the phenomenon to be studied become comparable to the gyroradii of the particles we are interested in, e.g.
- details of shocks (at the end of the course)
- relatively small or nonmagnetic objects in the solar wind (Mercury, Venus, Mars)

What options do we have?
- Vlasov theory
- Quasi-neutral hybrid approach

In QN approach electrons are still considered as a fluid (small gyroradii) but ions are described as (macro) particles

Quasi-neutrality requires that we are still in the plasma domain (larger scales than $\lambda_{De}$)
As electrons and ions are separated, Ohm’s law is important. Recall the generalized Ohm’s law

\[
E + \mathbf{V} \times \mathbf{B} = \frac{\mathbf{J}}{\sigma} + \frac{1}{n_e} \mathbf{J} \times \mathbf{B} - \frac{1}{ne^2} \nabla \cdot \mathbf{P}_e + \frac{m_e}{ne^2} \frac{\partial \mathbf{J}}{\partial t}
\]

\[-\mathbf{V}_e \times \mathbf{B} \quad \text{(similar to Hall MHD)}\]

\[\Rightarrow \]

\[
E = -\mathbf{V} \times \mathbf{B} + \frac{\nabla P_e}{ne^2} + \frac{\mathbf{J}}{\sigma}
\]

Inclusion of pressure term requires assumptions of adiabatic/isothermal behavior of electron fluid & \(T_e\).

Note: The QN equations cannot be casted to conservation form.

\[
\begin{align*}
n_e &= \left| q_e \right|^{-1} \sum_i q_i n_i \\
\mathbf{J} &= \sum_i q_i n_i \mathbf{V}_i + q_e n_e \mathbf{V}_e \\
\frac{d\mathbf{V}_i}{dt} &= \mathbf{v}_i \\
\frac{d\mathbf{V}_e}{dt} &= \frac{q_i}{m_i} (\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) \\
\mathbf{E} &= -\mathbf{V} \times \mathbf{B} + \frac{\nabla P_e}{ne^2} + \frac{\mathbf{J}}{\sigma} \\
\nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J}
\end{align*}
\]

(a) Mercury. Example of a magnetospheric interaction case with an Earth-like global intrinsic magnetic field.

(b) Venus. The dense atmosphere and the ionosphere of the planet interact with the solar wind. No intrinsic magnetic field.

(c) Mars. Venus-like case of ionospheric interaction.

(d) The Moon. Unmagnetized body with no atmosphere.
Quasi-neutral hybrid simulations of Titan’s interaction with the magnetosphere of Saturn during a flyby of the Cassini spacecraft 26 December 2005

Escape of O⁺ and H⁺ from the atmosphere of Venus.
Using the quasi-neutral hybrid simulation

Figure 1. Flow vectors of the planetary (top) O⁺ and (bottom) H⁺ ions in the regions where their bulk fluxes are high \((n_i > 10^{10} \text{ cm}^{-2} \text{ s}^{-1})\). Three different projections onto the \((x, y, z)\) and \((y, z)\) planes are shown. Vectors are normalized and their orientation and coloring illustrate the velocity. The background coloring shows the densities at the \(y = 0\), \(z = 0\) and \(x = 2R\), planes. The arrows in the lower left corners show the orientation of the velocity \((\mathbf{V}_i)\), electric field \((\mathbf{E}_{SB})\) and magnetic field \((\mathbf{B}_{SB})\) of the solar wind.
Beyond MHD: Kinetic Alfvén waves

At short wavelengths kinetic effects start to modify the Alfvén waves. Easier than to immediately go to Vlasov theory, is to look for kinetic corrections to the MHD dispersion equation.

For relatively large beta ($\beta > m_e/m_i$), e.g., in the solar wind, at magnetospheric boundaries, magnetotail plasma sheet the mode is called oblique kinetic Alfvén wave and it has the phase velocity

$$
\nu_\parallel = v_A \left[ 1 + k_\perp^2 r_A^2 \left( \frac{3}{4} + \frac{T_e}{T_i} \right) \right]^{1/2}
$$

$$
\nu_\perp = \frac{k_\perp v_A}{k_i} \left[ 1 + k_\perp^2 r_A^2 \left( \frac{3}{4} + \frac{T_e}{T_i} \right) \right]^{1/2}
$$

For smaller beta ($\beta \ll m_e/m_i$), e.g., above the auroral region and in the outer magnetosphere, the electron thermal speed is smaller than the Alfvén speed, electron inertial becomes important.

$$
\omega^2 = k_\parallel^2 v_A^2 \frac{1 + k_\perp^2 r_A^2}{1 + k_\perp^2 v_A^2/\omega_{pe}^2}
$$

inertial (kinetic) Alfvén wave or shear kinetic Alfvén wave.

Kinetic Alfvén waves

- carry field-aligned currents (important in M-I coupling)
- are affected by Landau damping (although small as long as $\beta$ is small), recall the Vlasov theory result

$$
\omega_i = -\frac{\omega_{pe}^2}{k_i} \frac{1}{1 + c^2/\nu_A^2} \sqrt{\frac{\pi m_i}{8k_B T_i}} \exp \left( -\frac{B^2}{2\mu_0 m_e k_B T_i} \frac{\omega_{ci}^2}{\omega_i^2} \right)
$$

Full Vlasov treatment leads to the dispersion equation

$$
\left( \frac{\omega}{k_\parallel v_A} \right)^2 = \frac{\mu_i \rho_i}{1 - \Gamma_0(\mu_i) + \Gamma_0(\mu_e)[1 + \xi Z(\xi)]}
$$

$$
\mu_i = k_\parallel^2 r_A^2
$$

$$
\rho_i = k_\parallel^2 \rho_e
$$

modified Bessel function of the first kind

$\Gamma_0(\mu) = \exp(-\mu) I_0(\mu)$

$\xi = \omega/k_\parallel v_A$

$\rho_e = \sqrt{2T_e/m_e}$