

Advanced Space Physics, 2012

Exercise set 3

Solutions to be presented on Monday, February 6, 2012

1. This is a little exercise to apply Hamiltonian mechanics in the electromagnetic field. It may require that you refresh your memory from your intermediate classical mechanics course.

The Lagrangian for a non-relativistic particle in cylindrical coordinates is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + qr\dot{\phi}A_\phi,$$

where no motion along z is assumed, and the magnetic field is assumed to be uniform and symmetric around the cylinder axis

Now the vector potential has the azimuthal component A_ϕ only. Show that it is $A_\phi = B(t)r/2$. Derive expressions for momenta p_r and p_ϕ and show that p_ϕ is a constant of motion.

Let $B(t)$ vary slowly in time and assume that $r \approx \text{constant}$ and $\dot{\phi} = -\epsilon\Omega$ (ϵ a small factor and Ω is the gyrofrequency). Now the polar angle is given by

$$\phi(t) = -\epsilon(\Omega(t)t + \kappa(t)),$$

where $\kappa(t)$ varies slowly with time. Calculate the adiabatic invariant

$$J = \oint d\kappa \left(p_r \frac{\partial r}{\partial \kappa} + p_\phi \frac{\partial \phi}{\partial \kappa} \right).$$

How is this related to the perpendicular energy W_\perp and magnetic moment μ .

2. Show that the *Harris model* can be derived from the vector potential

$$A_y(x, z) = -B_0F(z) + B_nx$$

and determine the functional form of F . Find a *cyclic coordinate* and the corresponding conserved quantity in this model (again, recall these concepts from Hamiltonian mechanics). Show further that particles move in the effective potential

$$U(x, z) = \frac{1}{2m}[P_y - qA_y(x, z)]^2,$$

where P_y is the momentum in the y -direction.

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3. Show that in the one-dimensional Harris model the equation of motion in the z -direction (i.e., perpendicular to the current sheet) is

$$\dot{z}^2 = (1 - k^2 + k^2 z^2)(1 - z^2).$$

Show that if an electric field in the y -direction (i.e., perpendicular to the magnetic field) is added, the equation of motion in the z -direction can be approximated for large t as

$$\ddot{z} \approx - \left(\frac{q}{m} \right)^2 \left(\frac{E_0 B_0}{L} \right) z t.$$

Find the solution of this equation in terms of Bessel functions using, e.g., mathematical tables and show that

$$z \approx - \frac{t^{-1/4}}{(E_0 B_0 / L)^{1/12} (q/m)^{1/6}} \times \left\{ A \cos \left[\frac{2}{3} \left(\frac{q}{m} \right) \left(\frac{B_0 E_0}{L} \right)^{1/2} t^{3/2} \right] + B \sin \left[\frac{2}{3} \left(\frac{q}{m} \right) \left(\frac{B_0 E_0}{L} \right)^{1/2} t^{3/2} \right] \right\}$$

is a reasonable approximation.

4. Consider a charged particle in a homogeneous magnetic field $\mathbf{B} = B_0 \mathbf{e}_z$ and an inhomogeneous electric field $\mathbf{E} = E_0 \cos(ky) \mathbf{e}_x$ where B_0 and E_0 are constant and $kr_L \ll 1$. Show that the inhomogeneity introduces a small correction to the $\mathbf{E} \times \mathbf{B}$ drift:

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} (1 - k^2 r_L^2 / 4).$$

What is the physical interpretation of the condition $kr_L \ll 1$. Compare the validity of this condition to electrons and protons.

Extra exercise (1) with extra bonus

Derive *Størmer's cut-off rigidity formula* in the dipole field

$$R_c = M \frac{\cos^4 \lambda}{r^2 [1 + (1 - \sin \epsilon \sin \phi \cos^3 \lambda)^{1/2}]^2},$$

where M is the magnetic moment in the unit system used by Størmer, λ is the geomagnetic latitude, ϵ the zenith angle, and ϕ the azimuthal angle measured from the direction of geomagnetic north with respect to geographic north.

Your teacher vaguely remembers trying to derive this result some 30 years ago but did not reach to the end. If you succeed, please send your calculation to Heli Hietala. The problem will be open as long as someone presents the solution. In that case it will be discussed at some later exercise session.

Exercise sets 4 and 5

Solutions to be presented on Monday (Feb 13) and Thursday (Feb 16), 2012

Warning: During week 7 two exercise session will take place!

1. Consider a one-dimensional, almost monochromatic wave packet that at the time $t = 0$ has the shape $u(x, 0) = f(x) \exp(ik_0x)$ (k_0 is real). Assume that $f(x)$ is of the form

$$f(x) = N \exp\left(-\frac{\alpha^2 x^2}{4}\right)$$

where N and α are real numbers. Calculate the Fourier components $A(k)$ of u in the k -space. Using your favourite computing tool (e.g. Matlab) sketch the functions $u(x, 0)$, $A(k)$, ja $|u(x, 0)|$ (extra point if done using a computer). Normalize the function $u(x, 0)$.

Define $(\Delta x)^2 = \langle x^2 - \langle x \rangle^2 \rangle$, where $\langle \rangle$ means

$$\langle x \rangle = \int_{-\infty}^{+\infty} u(x, 0)^* x u(x, 0) dx .$$

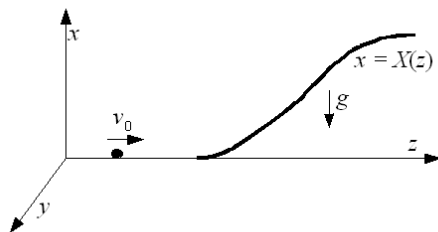
Calculate Δx , Δk , $\Delta x \Delta k$. Notice and explain the relationship to a similar product $(\Delta x \Delta k)$ in quantum mechanics.

2. Consider the wave packet of the previous exercise and assume that it propagates in a dispersive medium whose dispersion equation is

$$\omega(k) = \nu \left(1 + \frac{a^2 k^2}{2} \right)$$

where the frequency ν is constant. Show that a wave packet containing several wave components (i.e., a wide packet) maintains its form quite well whereas a narrow packet widens rapidly.

3. Consider the mechanical analogue for radio wave propagation to the ionosphere illustrated in the figure below. Launch a ball (mass m) rolling without friction



into the z -direction with the initial speed v_0 . When the ball reaches the rising surface $x = X(z)$, it is influenced by gravitation g in the direction of negative x -axis. Derive the expression for the velocity starting from the conservation

of total energy. What is the expression for the time when the ball returns to the starting point? What is the expression for the virtual reflection distance (assume $ds \approx dz$)? Compare these expressions to the expressions for the reflection of radio waves from the ionosphere. What quantity in this analogue corresponds to the frequency, plasma frequency and ionospheric density? How does the path of the ball change if the angle of the initial velocity and the z -axis in the yz -plane is $\theta_0 \neq 0$? Compare also this to the ionospheric propagation. What would the addition of friction correspond to in the ionosphere?

4. Assume that the ionospheric density profile is linear. Derive the expression for the distance between the radio wave transmitter and receiver for a signal reflected from the ionosphere as a function of frequency and transmission angle. Show that for certain distances and frequencies the receiver can be reached through three different ray paths (i.e., using three different transmission directions).
5. Show that the propagation time for an electromagnetic wave, with a frequency well above the gyro and plasma frequencies, from a distant source is

$$T = \int_0^d \frac{ds}{v_g} = \frac{d}{c} + Df^{-2},$$

where d is the distance to the source and D is the *dispersion measure*

$$D(\text{MHz}) = 4.07 \cdot 10^3 \int_0^d n_e(\text{cm}^{-3}) ds(\text{pc}).$$

Here pc is the astronomical distance measure *parsec*. If you do not know its definition, find it.

Hint: Start by deriving an approximation for v_g (or $1/v_g$) from the cold plasma dispersion equation at high frequencies.

6. Show that in electrostatic approximation ($\nabla \times \mathbf{E} = 0$, i.e., $\mathbf{k} \parallel \mathbf{E}$) the dispersion equation can be written as

$$\mathbf{k} \cdot \mathcal{K} \cdot \mathbf{k} = 0.$$

Using this result show that close to the lower-hybrid frequency there is a wave mode (the lower-hybrid wave) whose dispersion equation can be approximated as

$$\omega^2 = \omega_{LH}^2 \left(1 + \frac{m_i}{m_e} \frac{k_{\parallel}^2}{k_{\perp}^2} \right),$$

where the lower-hybrid frequency is

$$\omega_{LH}^2 = \frac{\omega_{ci}^2 + \omega_{pi}^2}{1 + (\omega_{pe}^2/\omega_{ce}^2)}.$$

Show further that ω_{LH} approaches the ion gyrofrequency ω_{ci} at the limit of low plasma density and the geometric mean of the electron and ion gyrofrequencies $\sqrt{\omega_{ce}\omega_{ci}}$ at the limit of large plasma density. Sketch the density dependence of the lower-hybrid frequency in the $\log n_e - \log \omega$ plane and draw the ion plasma frequency in the same picture (Recommendation: use a computer to produce the sketch).

Extra exercise (2) with extra bonus

Perform the detailed calculations leading to the electric field of a radio wave propagating vertically into a weakly inhomogeneous ionosphere in the WKB-approximation

$$\begin{aligned} E_x &= \frac{2E_0}{\sqrt{n}} \cos\left(k_0 \int_z^{z_0} n dz' + \frac{\pi}{4}\right) \exp(-i\omega t) \\ &= \frac{E_0}{\sqrt{n}} \left\{ \exp\left[\frac{i\pi}{4} + i\left(k_0 \int_z^{z_0} n dz' - \omega t\right)\right] \right. \\ &\quad \left. + \exp\left[\frac{-i\pi}{4} + i\left(-k_0 \int_z^{z_0} n dz' - \omega t\right)\right] \right\} . \end{aligned}$$

This is a problem for those who are fond of non-trivial differential equations and complex analysis. The problem will be open as long as someone presents the solution. In that case it will be discussed at some later exercise session.