Advanced Space Physics, 2012

During week 7 there will be two exercise sessions.

Exercise set 4

Solutions to be presented on Monday, February 13, 2012

1. Consider a one-dimensional, almost monochromatic wave packet that at the time $t = 0$ has the shape $u(x, 0) = f(x)\exp(ik_0x)$ (k_0 is real). Assume that $f(x)$ is of the form

$$f(x) = N \exp\left(-\frac{\alpha^2 x^2}{4}\right)$$

where N and α are real numbers. Calculate the Fourier components $A(k)$ of u in the k-space. Using your favourite computing tool (e.g. Matlab) sketch the functions $u(x, 0)$, $A(k)$, ja $|u(x, 0)|$ (extra point if done using a computer). Normalize the function $u(x, 0)$.

Define $(\triangle x)^2 = \langle x^2 - \langle x \rangle^2 \rangle$, where $\langle\ \rangle$ means

$$\langle x \rangle = \int_{-\infty}^{+\infty} u(x, 0)^* \, x \, u(x, 0) dx \,.$$

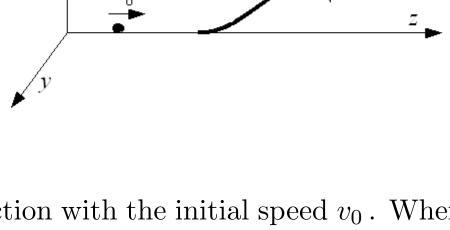
Calculate $\triangle x$, $\triangle k$, $\triangle x \triangle k$. Notice and explain the relationship to a similar product ($\triangle x \triangle k$) in quantum mechanics.

2. Consider the wave packet of the previous exercise and assume that it propagates in a dispersive medium whose dispersion equation is

$$\omega(k) = \nu \left(1 + \frac{a^2 k^2}{2}\right)$$

where the frequency ν is constant. Show that a wide wave packet maintains its form quite well whereas a narrow packet widens rapidly.

3. Consider the mechanical analogue for radio wave propagation to the ionosphere illustrated in the figure below. Launch a ball (mass m) rolling without friction



into the z-direction with the initial speed v_0. When the ball reaches the rising surface $x = X(z)$, it is influenced by gravitation g in the direction of negative

x-axis. Derive the expression for the velocity starting from the conservation of total energy. What is the expression for the time when the ball returns to the starting point? What is the expression for the virtual reflection distance (assume $ds \approx dz$)? Compare these expressions to the expressions for the reflection of radio waves from the ionosphere. What quantity in this analogue corresponds to the frequency, plasma frequency and ionospheric density? How does the path of the ball change if the angle of the initial velocity and the z-axis in the yz-plane is $\theta_0 \neq 0$? Compare also this to the ionospheric propagation. What would the addition of friction correspond to in the ionosphere?

Extra exercise (2) with extra bonus

Perform the detailed calculations leading to the electric field of a radio wave propagating vertically into a weakly inhomogeneous ionosphere in the WKB-approximation

$$\begin{aligned} E_x &= \frac{2E_0}{\sqrt{n}} \cos\left(k_0 \int_z^{z_0} n\, dz' + \frac{\pi}{4}\right) \exp(-i\omega t) \\ &= \frac{E_0}{\sqrt{n}} \left\{ \exp\left[\frac{i\pi}{4} + i\left(k_0 \int_z^{z_0} n\, dz' - \omega t\right)\right] \right. \\ &\quad \left. + \exp\left[\frac{-i\pi}{4} + i\left(-k_0 \int_z^{z_0} n\, dz' - \omega t\right)\right] \right\} . \end{aligned}$$

This is a problem for those who are fond of non-trivial differential equations and complex analysis. The problem will be open as long as someone presents the solution. In that case it will be discussed at some later exercise session.

Exercise set 5

Solutions to be presented on Thursday, February 16, 2012

1. Assume that the ionospheric density profile is linear. Derive the expression for the distance between the radio wave transmitter and receiver for a signal reflected from the ionosphere as a function of frequency and transmission angle. Show that for certain distances and frequencies the receiver can be reached through three different ray paths (i.e., using three different transmission directions).

2. Show that the propagation time for an electromagnetic wave, with a frequency well above the gyro and plasma frequencies, from a distant source is

$$T = \int_0^d \frac{ds}{v_g} = \frac{d}{c} + Df^{-2}\,,$$

where d is the distance to the source and D is the *dispersion measure*

$$D(\mathrm{MHz}) = 4.07 \cdot 10^3 \int_0^d n_e(\mathrm{cm}^{-3})\, ds(\mathrm{pc})\,.$$

Here pc is the astronomical distance measure *parsec*. If you do not know its definition, find it.

Hint: Start by deriving an approximation for v_g (or $1/v_g$) from the cold plasma dispersion equation at high frequencies.

3. Show that in electrostatic approximation ($\nabla \times \mathbf{E} = 0$, i.e., $\mathbf{k} \parallel \mathbf{E}$) the dispersion equation can be written as

$$\mathbf{k} \cdot \mathcal{K} \cdot \mathbf{k} = 0\,.$$

Using this result show that close to the lower-hybrid frequency there is a wave mode (the lower-hybrid wave) whose dispersion equation can be approximated as

$$\omega^2 = \omega_{LH}^2 \left(1 + \frac{m_i}{m_e}\frac{k_\parallel^2}{k_\perp^2}\right)\,,$$

where the lower-hybrid frequency is

$$\omega_{LH}^2 = \frac{\omega_{ci}^2 + \omega_{pi}^2}{1 + (\omega_{pe}^2/\omega_{ce}^2)}\,.$$

Show further that ω_{LH} approaches the ion gyrofrequency ω_{ci} at the limit of low plasma density and the geometric mean of the electron and ion gyrofrequencies $\sqrt{\omega_{ce}\omega_{ci}}$ at the limit of large plasma density. Sketch the density dependence of the lower-hybrid frequency in the $\log n_e - \log \omega$ plane and draw the ion plasma frequency in the same picture (Recommendation: use a computer to produce the sketch).

Exercise set 6 (preliminary version)

Solutions to be presented on Thursday, February 23.

1. A less rigorous procedure than Landau's method to solve the Vlasov equation is to start from the result obtained first by Vlasov himself

$$1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \int \frac{\mathbf{k} \cdot \partial f_{\alpha 0}/\partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \, d^3v = 0 \, .$$

   Add weak collisions in the Vlasov equation in the form $\partial f/\partial t|_c = -\nu(f - f_0)$ and show that the Fourier transform method leads to the Landau prescription at the limit $\nu \to 0^+$.

2. (a) Show that the plasma dispersion function

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-x^2)}{x - \zeta} \, dx \quad ; \quad Im(\zeta) > 0$$

   fulfils the equation

$$Z'(\zeta) = -2[1 + \zeta Z(\zeta)]$$

   (b) Find (e.g., using literature) the series expansions of $Z(\zeta)$ for large and small arguments.

3. Show that for Maxwellian electrons (and fixed background ions) the dispersion equation can be written in the form

$$1 - \frac{\omega_{pe}^2}{k^2 v_{the}^2} Z' \left(\frac{\omega}{k v_{the}} \right) = 0$$

   where Z is the plasma dispersion function.

4. Assume that $T_e \gg T_i$ and look for solutions of the electrostatic dispersion equation in the phase velocity range

$$\sqrt{\frac{k_B T_i}{m_i}} < \frac{\omega}{k} < \sqrt{\frac{k_B T_e}{m_e}} \, .$$

   That is derive the frequency for the ion–acoustic wave

$$\omega_r^2 = \frac{k^2 c_s^2}{1 + k^2 \lambda_{De}^2} \quad ; \quad c_s = \sqrt{\frac{k_B T_e}{m_i}}$$

   and its damping rate

$$\omega_i = -\frac{|\omega_r| \sqrt{\pi/8}}{(1 + k^2 \lambda_{De}^2)^{3/2}} \left[\left(\frac{T_e}{T_i} \right)^{3/2} \exp\left(\frac{-T_e/T_i}{2(1 + k^2 \lambda_{De}^2)} \right) + \sqrt{\frac{m_e}{m_i}} \right] \, .$$

   Estimate for what temperature ratio T_e/T_i the damping rate approaches the wave frequency $|\omega_i| \to \omega_r$?