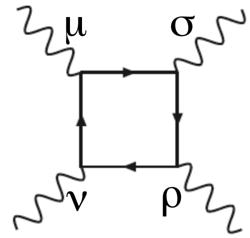
Session of Monday 17 November at 16.00-17.30 in aud A315.

1. Kubis p. I.12 gives the effective low-energy lagrangian corresponding to the light-by-light scattering diagram below. Show that this loop diagram $L^{\mu\nu\rho\sigma}$ (which is symmetric in the Lorentz indices μ, ν, ρ, σ of the interaction vertices) is UV finite. This is required by renormalizability since QED has no 4-photon coupling.

   Hint: The only potentially divergent contributions arise from the leading term in the loop momentum for all propagators. Consider the possible structure of Lorentz indices, then simplify the Dirac algebra by summing over some Lorentz indices.

2. (a) Derive the expressions for the vector and axial vector Noether currents of the lagrangian $\mathcal{L} = \frac{F^2}{4}\left\langle(\partial_\mu U)\partial^\mu U^\dagger\right\rangle$ given on Kubis I.24:

$$V_a^\mu = i\frac{F^2}{4}\left\langle\lambda_a\left[\partial^\mu U, U^\dagger\right]\right\rangle \qquad\qquad A_a^\mu = i\frac{F^2}{4}\left\langle\lambda_a\left\{\partial^\mu U, U^\dagger\right\}\right\rangle$$

   (b) Using this result and $U = \exp(i\phi/F)$ show that to first order in ϕ,

$$\langle 0|A_a^\mu(x)|\phi_b(p)\rangle = ip^\mu e^{-ip\cdot x}\delta_{ab}F$$

3. The effective four-fermion interaction governing pion decay is

$$\mathcal{L}(x) = \frac{G_F}{\sqrt{2}} V_{ud}\, \bar{d}(x)\gamma_\alpha(1-\gamma_5)u(x)\, \bar{\nu}_\mu(x)\gamma^\alpha(1-\gamma_5)\mu(x) \tag{1}$$

   where the Fermi coupling $G_F = 1.16639\cdot 10^{-5}$ GeV^{-2}, the CKM matrix element $V_{ud} = 0.9753$ and the quark and lepton fields are as indicated.

   (a) Calculate the pion decay $\pi^+ \to \mu^+\nu_\mu$ matrix element $T = \langle\mu^+(k)\nu(q)|\int d^4x\mathcal{L}(x)|\pi^+(p)\rangle$ making use of

$$\langle 0|\bar{d}(x)\gamma^\mu\gamma_5 u(x)|\pi^+(p)\rangle = -i\sqrt{2}f_\pi p^\mu e^{-ip\cdot x} \tag{2}$$

   where the pion decay constant $f_\pi = 93$ MeV.

   (b) Using the matrix element and the expression for the decay width given in the Feynman rules of Peskin, derive

$$\Gamma(\pi^+ \to \mu^+\nu_\mu) = \frac{G_F^2 V_{ud}^2 f_\pi^2}{4\pi m_\pi^3}\, m_\mu^2(m_\pi^2 - m_\mu^2)^2 \tag{3}$$

   and compare with the experimental value of the π^+ life-time, $\tau_\pi = 2.60\cdot 10^{-8}$ s.

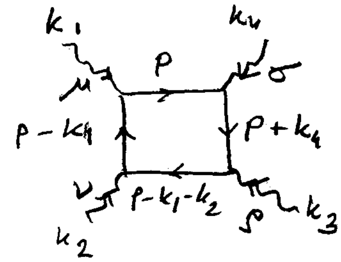
   (c) Calculate the ratio $\Gamma(\pi^+ \to e^+\nu_e)/\Gamma(\pi^+ \to \mu^+\nu_\mu)$, and compare with the experimental value $1.23\cdot 10^{-4}$. Explain the smallness of this ratio, *i.e.*, why the decay width vanishes with the charged lepton mass.

1.

QCD 08/11

$$iL^{\mu\nu\rho\sigma} = (-ie)^4 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\Big[S(p)\gamma^\mu S(p-k_1)\gamma^\nu$$

$$\cdot S(p-k_1-k_2)\gamma^\rho S(p+k_4)\gamma^\sigma\Big] +$$



+ 5 permutations of external lines

$\Rightarrow$ Fully symmetric under exchange $(\mu, k_1) \leftrightarrow (\nu, k_2) \leftrightarrow \dots$

$$S(p) = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} = \mathcal{O}\left(\frac{1}{p}\right) \quad \text{for } p \to \infty$$

$\Rightarrow \int \frac{d^4p}{(2\pi)^4}\, \mathcal{O}\left(\frac{1}{p^4}\right)$ potentially log divergent, so coefficient must vanish

$$L_D^{\mu\nu\rho\sigma} = -i \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}[\not{p}\gamma^\mu\not{p}\gamma^\nu\not{p}\gamma^\rho\not{p}\gamma^\sigma]}{(p^2)^4}$$

$$\propto g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}$$ | Fully symmetric, No external momenta

$$L_{D\,\mu}^{\mu}{}^{\rho}{}_{\rho} \propto 16 + 4 + 4 = 24 \neq 0$$ | $g^\mu{}_\mu = 4$

$$\mathrm{Tr}[\not{p}\underbrace{\gamma^\mu\not{p}\gamma_\mu}_{-2\not{p}}\not{p}\gamma^\rho\not{p}\gamma_\rho] = (-2)^2\,\mathrm{Tr}[\not{p}\not{p}\not{p}\not{p}] = 16(p^2)^2$$

$$\mathrm{Tr}[\not{p}\underbrace{\gamma^\mu\not{p}\gamma^\rho\not{p}\gamma_\mu}_{-2\not{p}\gamma^\rho\not{p}}\not{p}\gamma_\rho] = -2\,\mathrm{Tr}[\not{p}\not{p}\underbrace{\gamma^\rho\not{p}\not{p}\gamma_\rho}_{4p^2}] = -32\,(p^2)^2$$

Of the 6 permutations, 4 have neighboring vertices μ, ν while 2 have μ, ν diagonally opposite

$$\Rightarrow 4 \cdot 16 + 2 \cdot (-32) = 0$$

| In QCD the permutations are weighted by color factors $\Rightarrow$ No cancellation!

2. Recall Noether's theorem : Peskin & Schroeder

If $S(\phi) = \int d^4x\, \mathcal{L}(\phi)$ is symmetric under $\phi \to \phi + \alpha \Delta\phi$
$(\alpha \neq \alpha(x))$

then $\mathcal{L}(\phi) \to \mathcal{L}(\phi) + \alpha \partial_\mu \mathcal{J}^\mu$ (change by divergence)

Also,

$$\alpha \Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}(\alpha \Delta\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\alpha \Delta \phi) =$$

$$= \alpha \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right] + \alpha \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \right]}_{=0 \text{ (eq. of motion)}} \Delta\phi$$

$$\Rightarrow j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \Delta\phi - \mathcal{J}^\mu \quad \text{satisfies } \partial_\mu j^\mu = 0$$

$\mathcal{L}^{(2)} = \frac{1}{4} F^2 \mathrm{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right]$ is symmetric under:

$$U \to LU : \quad \mathcal{L}^{(2)} \to \frac{1}{4} F^2 \mathrm{Tr}\left[L \partial_\mu U (\partial^\mu U^\dagger) L^\dagger\right] = \mathcal{L}^{(2)}$$

$$U \to UR^\dagger : \quad \mathcal{L}^{(2)} \to \frac{1}{4} F^2 \mathrm{Tr}\left[(\partial_\mu U) R^\dagger R\, \partial^\mu U^\dagger\right] = \mathcal{L}^{(2)}$$

$\Rightarrow \mathcal{J}^\mu = 0$: Not only S, but $\mathcal{L}$ itself has symmetry.

$L = 1 - i\alpha_L^a \lambda_a$ $\qquad \lambda_a$: SU(3) generators (Gell-Mann matrices)

$\Delta U = -i\alpha_L^a \lambda_a U \qquad \Delta U^\dagger = +i\alpha_L^a U^\dagger \lambda_a$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu U)} \Delta U + \frac{\partial \mathcal{L}}{\partial(\partial_\mu U^\dagger)} \Delta U^\dagger = \frac{-i\alpha_L^a}{4} F^2 \mathrm{Tr}\{\lambda_a U \partial^\mu U^\dagger - (\partial^\mu U) U^\dagger \lambda_a\}$$

$$= +\frac{i\alpha_L^a}{4} F^2 \mathrm{Tr}\{2\lambda_a (\partial^\mu U) U^\dagger - \lambda_a \underbrace{\partial^\mu (U U^\dagger)}_{=0}\}$$

2 cont) $L^\mu_a = \frac{i}{4} F^2 \operatorname{Tr}\{\lambda_a (\partial^\mu U) U^+\}$

$R = 1 - i\alpha^a_R \lambda_a \qquad \Delta U = i\alpha^a_R U \lambda_a , \quad \Delta U^+ = -i\alpha^a_R \lambda_a U^+$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu U)} \Delta U + \frac{\partial \mathcal{L}}{\partial(\partial_\mu U^+)} \Delta U^+ = \frac{i\alpha^a_R}{4} F^2 \operatorname{Tr}\{U \lambda_a \partial^\mu U^+ - (\partial^\mu U)\lambda_a U^+\}$$

$$= -\frac{i\alpha^a_R}{4} F^2 \operatorname{Tr}\{2 \, (\partial^\mu U) \lambda_a U^+\}$$

$$R^\mu_a = -\frac{i}{4} F^2 \operatorname{Tr}\{(\partial^\mu U)\lambda_a U^+\}$$

$$V^\mu_a = R^\mu_a + L^\mu_a = \frac{i}{4} F^2 \operatorname{Tr}\{\lambda_a [\partial^\mu U, U^+]\}$$

$$-A^\mu_a = R^\mu_a - L^\mu_a = -\frac{i}{4} F^2 \operatorname{Tr}\{\lambda_a \{\partial^\mu U, U^+\}\}$$ | change of sign in def of A

The sign and normalization of the conserved currents being arbitrary, the definition is motivated by the expansion in ϕ : $U = \exp(i\phi/F) = 1 + i\phi/F + \dots$

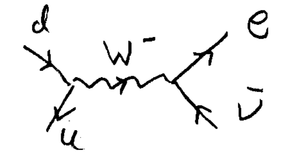
$$A^\mu_a = \frac{i}{4} F^2 \operatorname{Tr}\{\lambda_a \; \{\tfrac{i}{F}\partial^\mu \phi , \; 1 - \tfrac{i}{F}\phi^+\}\}$$

$$= -\frac{1}{2} F \underbrace{\operatorname{Tr}\{\lambda_a \lambda_b\}}_{2\delta_{ab}} \partial^\mu \phi_b = -F \partial^\mu \phi_a$$

$$\langle 0 | A^\mu(x) | \phi_b(p)\rangle = e^{-ip\cdot x} \langle 0 | A^\mu(0) | \phi_b(p)\rangle$$

$$= -F \partial_\mu \langle 0 | \phi_a(x) | \phi_b(p)\rangle = i p_\mu e^{-ip\cdot x} \delta_{ab} F$$

3a) $\mathcal{L}_F(x) = \frac{G_F}{\sqrt{2}} V_{ud}\, \bar{d} \gamma_\mu (1-\gamma_5) u \; \bar{\nu} \gamma^\mu (1-\gamma_5) e$



$V_{ud} = .9753$

$G_F = 1.16639(1) \cdot 10^{-5}\ \mathrm{GeV}^{-2}$ $\qquad f_\pi = 93\ \mathrm{MeV}$

$$\langle 0 | \bar{d}(x) \gamma^\mu \gamma_5 u(x) | \pi^+(p) \rangle = -i\sqrt{2}\, f_\pi\, p^\mu e^{-i p \cdot x}$$

$$\langle \mu^+(k) \nu_\mu(q) | \bar{\nu}(x) \gamma^\mu (1-\gamma_5) e(x) | 0 \rangle = \bar{u}(q) \gamma^\mu (1-\gamma_5) v(k)\, e^{i(k+q)\cdot x}$$

$$T = \langle \mu^+(k) \nu_\mu(q) | \int d^4x\, \mathcal{L}_F(x) | \pi^+(p) \rangle = (2\pi)^4 \delta^4(p-k-q)$$

$$* \; \frac{G_F}{\sqrt{2}} V_{ud} (-i\sqrt{2}) f_\pi \; \bar{u}(q) \not{p} (1-\gamma_5) v(k)$$

3b) $i\mathcal{M} = G_F V_{ud} f_\pi \bar{u}(q) \not{p} (1-\gamma_5) v(k)$

$$\Gamma = \frac{1}{2m_\pi} \frac{1}{16\pi^2} \frac{p_{cm}}{m_\pi} \int d\Omega\, \overline{|\mathcal{M}|^2} \qquad \text{Peskin (A, 58)}$$

$$p_{cm} = \frac{1}{2m_\pi} \sqrt{\lambda(m_\pi^2, m_\mu^2, 0)} = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

$$|\overline{\mathcal{M}}|^2 = G_F^2 V_{ud}^2 f_\pi^2 \, \mathrm{Tr}\left[(\not{q} + \overset{0}{m_e}) \not{p} (1-\gamma_5) \not{k} (1+\gamma_5) \not{p}\right]$$

$$\mathrm{Tr} = 2\, \mathrm{Tr}\left[\not{q} \not{p} \overbrace{\not{k}}^{\not{p}-\not{q}} (1+\overset{0}{\gamma_5}) \not{p}\right] = 2\, \mathrm{Tr}\left[m_\pi^2 \not{q} \not{p} - 2p\cdot q\, \not{q} \not{p}\right]$$

$$k = p - q; \quad m_\mu^2 = m_\pi^2 - 2p\cdot q; \quad 2p\cdot q = m_\pi^2 - m_\mu^2$$

$$\mathrm{Tr} = 2 m_\mu^2\, \mathrm{Tr}(\not{p}\not{q}) = 4 m_\mu^2\, 2p\cdot q = 4 m_\mu^2 (m_\pi^2 - m_\mu^2)$$

$$\overline{|\mathcal{M}|^2} = 4 G_F^2 V_{ud}^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2) \qquad (\text{indep. of } \Omega)$$

3b cont.)

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{p_{CM}}{8\pi m_\pi^2} \overline{|\mathcal{M}|^2} =$$

$$= \frac{G_F^2 V_{ud}^2 f_\pi^2}{4\pi m_\pi^3} m_\mu^2 (m_\pi^2 - m_\mu^2)^2$$

$$= \frac{\{1.16639 \cdot 0.9753 \cdot 0.093 \cdot 0.106 \cdot [(0.140)^2 - (0.106)^2]\}^2}{4\pi (0.140)^3} \cdot 10^{-10}$$

$$= 2.55 \cdot 10^{-17} \text{ GeV}$$

$$\tau(\pi^+ \to \mu^+ \nu_\mu) = \frac{1}{\Gamma_{\mu\nu}} = \frac{6.582 \cdot 10^{-25} \text{ GeV} \cdot s}{2.55 \cdot 10^{-17} \text{ GeV}} = 2.58 \cdot 10^{-8} s$$

$$(\tau_{exp} = 2.60 \cdot 10^{-8} s)$$

3c)

$$\frac{\Gamma(\pi^+ \to e^+ \nu_e)}{\Gamma(\pi^+ \to \mu^+ \nu_\mu)} = \left(\frac{m_e}{m_\mu} \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \right)^2 = 1.28 \cdot 10^{-4}$$

$$= 1.23 \cdot 10^{-4} \text{ (exp.)}$$

$\Gamma(\pi \to \bar{\ell} \nu_\ell) = 0$ if $m_\ell = 0$ due to V-A coupling, which allows only lefthanded leptons and right-handed antileptons in the massless limit

$$\nu_\ell \xleftarrow{\overset{L}{\Rightarrow}} \pi \xrightarrow{\overset{R}{\Rightarrow}} \bar{\ell}$$

Spin component +1 along $\bar{\ell}$ direction, whereas π is spinless