Session of Monday 24 November at 16.00-17.30 in aud A315.

1. (a) Verify the statement on Kubis p. II.10 that the ϕ_3 and ϕ_8 fields in the chiral lagrangian mix according to

$$\begin{aligned} \mathcal{L}^{(2)} &= \frac{F^2}{4}\mathrm{Tr}\left[\partial_\mu U \partial^\mu U^\dagger + 2B(\mathcal{M}U^\dagger + \mathcal{M}^\dagger U)\right] \\ &\supset -\frac{1}{2}B\begin{bmatrix}\phi_3 & \phi_8\end{bmatrix}\begin{bmatrix} m_u + m_d & (m_u - m_d)/\sqrt{3} \\ (m_u - m_d)/\sqrt{3} & (m_u + m_d + 4m_s)/3\end{bmatrix}\begin{bmatrix}\phi_3 \\ \phi_8\end{bmatrix}\end{aligned}$$

   (b) Show that the eigenvalues of the matrix in (a) determine the masses $m_{\pi^0}^2 = B(m_u + m_d)$ and $m_\eta^2 = B(m_u + m_d + 4m_s)/3$, up to corrections of $\mathcal{O}\left((m_u - m_d)^2\right)$.

2. (a) Show using the classical equations of motion of the QCD lagrangian with a single massless quark mass that $\partial_\mu j_5^\mu = 0$, where the axial vector current $j_5^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$. Find the expression for $\partial_\mu j_5^\mu$ also when the quark mass $m_q \neq 0$.

   (b) The Adler-Bell-Jackiw anomaly

$$\partial_\mu j_5^\mu = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}$$

   states that the axial vector current is not conserved at the operator (quantum) level even when the quark mass vanishes. Show that the anomaly implies

$$\Delta N_R - \Delta N_L = \frac{e^2}{2\pi^2}\int d^4x\, \boldsymbol{E}\cdot\boldsymbol{B}$$

   where $N_R = \int d^3\boldsymbol{x}\psi_R^\dagger\psi_R$ is the total number of right-handed fermions, and similarly for L. Δ refers to the change in number from $x^0 = -\infty$ to $x^0 = +\infty$, and $\boldsymbol{E}, \boldsymbol{B}$ are the electric and magnetic fields. [This is problem 19.1 of Peskin and Schroeder.]

3. The $\pi^0 \to \gamma + \gamma$ decay width can be calculated from the Bell-Jackiw anomaly as described in, *e.g.*, section 19.3 of Peskin and Schroeder. The $j_5^{\mu 3}$ component of the axial current, which can annihilate a π^0, has the electromagnetic anomaly

$$\partial_\mu j_5^{\mu 3} = -\frac{e^2}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \tag{1}$$

   where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the QED field strength tensor. The matrix element of the current between the vacuum and a state with 2 photons of momenta p and k (helicities not indicated explicitly) has the form

$$\langle p, k|j_5^{\mu 3}(q)|0\rangle = \varepsilon_\nu^*(p)\varepsilon_\lambda^*(k)\mathcal{M}^{\mu\nu\lambda}(p,k) \tag{2}$$

PTO

(a) Verify that the decomposition of the amplitude

$$\begin{aligned} \mathcal{M}^{\mu\nu\lambda}(p,k) &= q^\mu \epsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta \mathcal{M}_1 + (\epsilon^{\mu\nu\alpha\beta} k^\lambda - \epsilon^{\mu\lambda\alpha\beta} p^\nu) k_\alpha p_\beta \mathcal{M}_2 \\ &+ [(\epsilon^{\mu\nu\alpha\beta} p^\lambda - \epsilon^{\mu\lambda\alpha\beta} k^\nu) k_\alpha p_\beta - \epsilon^{\mu\nu\lambda\sigma} (p-k)_\sigma \, p\cdot k] \mathcal{M}_3 \end{aligned} \tag{3}$$

where the invariant amplitudes $\mathcal{M}_i = \mathcal{M}_i(q^2)$, is consistent with the constraints

(i) $p_\nu \mathcal{M}^{\mu\nu\lambda}(p,k) = k_\lambda \mathcal{M}^{\mu\nu\lambda}(p,k) = 0$ (gauge invariance);
(ii) $\mathcal{M}^{\mu\nu\lambda}(p,k) = \mathcal{M}^{\mu\lambda\nu}(k,p)$ (identical photons);
(iii) $\mathcal{M}^{\mu\nu\lambda}(p,k) = \mp \mathcal{M}^{\mu\nu\lambda}(k,p)$, with $-(+)$ for $\mu = 0$ ($\mu = 1,2,3$) (parity invariance in the $\boldsymbol{q} = 0$ frame since $\boldsymbol{p} = -\boldsymbol{k}$ and the polarization (axial) vectors do not change under parity).

(b) Show that (3) implies

$$iq_\mu \mathcal{M}^{\mu\nu\lambda}(p,k) = iq^2 \epsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta (\mathcal{M}_1 + \mathcal{M}_3) \tag{4}$$

and that using (1) one gets

$$iq_\mu \mathcal{M}^{\mu\nu\lambda}(p,k) = \langle p,k|\partial_\mu j_5^{\mu 3}(q)|0\rangle = -\frac{e^2}{4\pi^2} \epsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta \tag{5}$$

(c) Either $\mathcal{M}_1$ or $\mathcal{M}_3$ must have a $1/q^2$ pole for (4) and (5) to be compatible. Only the massless π^0 propagator in $\langle p,k|S|\pi^0\rangle\langle \pi^0|j_5^{\mu 3}(q)|0\rangle$ can give this pole. Using $\langle \pi^0|j_5^{\mu 3}(x)|0\rangle = ip^\mu f_\pi \exp(ip\cdot x)$ and writing the pion decay amplitude

$$i\mathcal{M}(\pi^0 \to 2\gamma) \equiv \langle p,k|S|\pi^0\rangle = iA\, \varepsilon_\nu^*(p) \varepsilon_\lambda^*(k)\, \epsilon^{\nu\lambda\alpha\beta} p_\alpha k_\beta \tag{6}$$

show that the pion pole appears in $\mathcal{M}_1$ as

$$\mathcal{M}_1 = -i\frac{f_\pi}{q^2} A = i\frac{e^2}{4\pi^2}\frac{1}{q^2} \tag{7}$$

(d) Recalling a factor 1/2 in the phase space integration due to the identical photons, and the expression for the decay width into two particles given in Peskin et al, show that

$$\Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2}{64\pi^3}\frac{m_\pi^3}{f_\pi^2} = \frac{1}{0.77\cdot 10^{-16}\ \mathrm{s}}$$

which is in good agreement with $\tau_{exp} = (0.84 \pm .06)\cdot 10^{-16}$ s.

Note: You can solve a later task without having done the previous ones.