1. $$\frac{d\alpha_s}{d\ln\mu^2} = -\beta_0 \alpha_s^2 \; ; \quad \beta_0 = \frac{33-2n_f}{12\pi} \overset{(n_f=3)}{=} \frac{27}{12\pi}$$

(a) $$d\left(\frac{1}{\alpha_s}\right) = \beta_0 \, d\ln\mu^2 ;$$

$$\alpha_s = \frac{1}{\beta_0 \ln\mu^2 + c} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)} = \frac{12\pi}{27\ln(\mu^2/\Lambda^2)}$$

(b) $\mu = m_Z = 91.2$ GeV : $\alpha_s(m_Z^2) = 0.119$

$$\ln\left(\frac{\mu^2}{\Lambda^2}\right) = \frac{12\pi}{27\alpha_s} \; , \quad \Lambda_{QCD} = m_Z \exp\left[-\frac{6\pi}{27\alpha_s}\right] = 258 \text{ MeV}$$

$$\alpha_s(9 \text{ GeV}^2) = \frac{6\pi}{27\ln(3/.258)} = 0.285$$

(c) $$\alpha_s(0) = \frac{6\pi}{27\ln(m_s/.258)} = 0.64$$

2.

a) $$S(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

$$S(t, \vec{p}) = \int \frac{dp^0}{2\pi} e^{-itp^0} S(p)$$

$$S(x^+, \vec{p}) = \int \frac{dp^-}{2\pi} e^{-x^+p^-/2} S(p)$$

$$S(t, \vec{p}) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dp^0 \frac{e^{-itp^0}}{(p^0 - E_p + i\varepsilon)(p^0 + E_p - i\varepsilon)}$$

$$= \Theta(t) \frac{i(-2\pi i)}{2\pi} \frac{e^{-itE_p}}{2E_p} + \Theta(-t) \frac{i\, 2\pi i}{2\pi} \frac{e^{iE_p t}}{-2E_p}$$

$$= \frac{1}{2E_p} \left[\Theta(t) e^{-iEt} + \Theta(-t) e^{iEt} \right]; \quad E = +\sqrt{\vec{p}^2 + m^2}$$

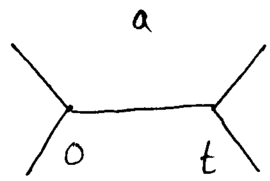
$$S(x^+, \vec{p}) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dp^- \frac{e^{-ix^+p^-/2}}{p^+p^- - p_\perp^2 - m^2 + i\varepsilon}$$

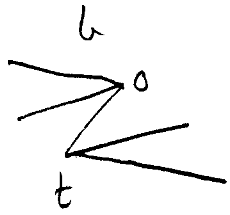
$$= \frac{i}{2\pi} \Big[\Theta(x^+) \frac{\Theta(p^+)}{p^+} (-2\pi i) e^{-ix^+p^-/2}$$

$$+ \Theta(-x^+) \frac{\Theta(-p^+)}{p^+} 2\pi i \, e^{-ix^+p^-/2} \Big]$$

$$= \Theta(x^+p^+) \frac{1}{|p^+|} e^{-ix^+p^-/2}, \quad p^- = \frac{p_\perp^2 + m^2}{p^+}$$

2b)





$$iM_a = (-ig)^2 \int_0^\infty dt\, S(t, \vec{p}_a + \vec{p}_b)\, e^{+i(E_c+E_d)t}$$

$$= \frac{-g^2}{2E_{ab}} \int_0^\infty dt\, e^{-i(E - E_c - E_d - i\varepsilon)t} = \frac{+ig^2}{2E_{ab}} \frac{1}{E - E_c - E_d - i\varepsilon}$$

$$iM_b = (-ig)^2 \int_{-\infty}^0 dt\, S(t, \vec{p}_a + \vec{p}_b)\, e^{i(E_c+E_d)t}$$

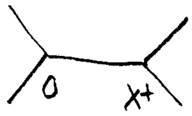
$$= -\frac{g^2}{2E_{ab}} \int_{-\infty}^0 dt\, e^{+i(E+E_c+E_d-i\varepsilon)t}$$

$$= \frac{ig^2}{2E_{ab}} \frac{1}{E+E_c+E_d-i\varepsilon}$$

$$M_a + M_b = \frac{g^2}{2E_{ab}} \frac{2E_{ab}}{E_{ab}^2 - (E_c+E_d)^2 - i\varepsilon} = \frac{g^2}{M^2 + (\vec{p}_a+\vec{p}_b)^2 - (p_c+p_d)^2 - (\vec{p}_c - \vec{p}_d)^2 - i\varepsilon}$$

$$= \frac{-g^2}{(p_c+p_d)^2 - M^2 + i\varepsilon}$$

2 c) $p^+_{ab} > 0$



$$iM = (-ig)^2 \frac{1}{2}\int_0^\infty dx^+ \, S(x^+, \vec{p}_a + \vec{p}_b)\, e^{ix^+(p_c^- + p_d^-)/2}$$

$$= \frac{-g^2}{2p^+_{ab}} \int_0^\infty dx^+ \, e^{-ix^+(p^-_{ab} - p^-_c - p^-_d - i\varepsilon)/2}$$

$$= \frac{ig^2}{p^+_{ab}} \, \frac{1}{p^-_{ab} - p_c^- - p_d^- - i\varepsilon}$$

$$(\vec{p}_\perp^{\,c} + \vec{p}_\perp^{\,d})^2 + M^2 - (p_c^+ + p_d^+)\left[\underbrace{\frac{p_{c\perp}^2 + m_c^2}{p_c^+}}_{p_c^-} + \underbrace{\frac{p_{d\perp}^2 + m_d^2}{p_d^+}}_{p_d^-}\right]$$

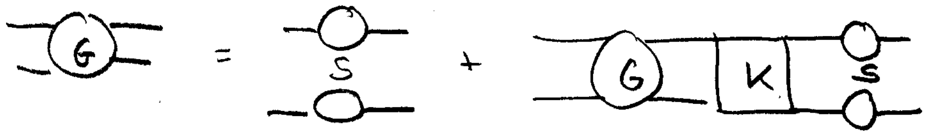
$$= -(p_c + p_d)^2 + M^2$$

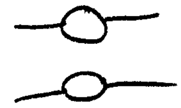
$$M = -\frac{g^2}{(p_a+p_b)^2 - M^2 + i\varepsilon} \qquad \text{ok.}$$

3) a)
$$\frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} = \frac{1}{2E} \sum_s \left[\frac{u(\vec{p},s)\,\bar{u}(\vec{p},s)}{p^0 - E + i\varepsilon} + \frac{v(-\vec{p},s)\,\bar{v}(-\vec{p},s)}{p^0 + E - i\varepsilon} \right]$$

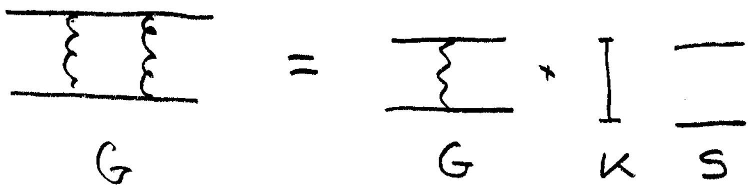
$$= \frac{1}{2E}\left[\frac{E\gamma^0 - \vec{p}\cdot\vec{\gamma} + m}{p^0 - E + i\varepsilon} + \frac{E\gamma^0 + \vec{p}\cdot\vec{\gamma} - m}{p^0 + E - i\varepsilon} \right] = \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}$$

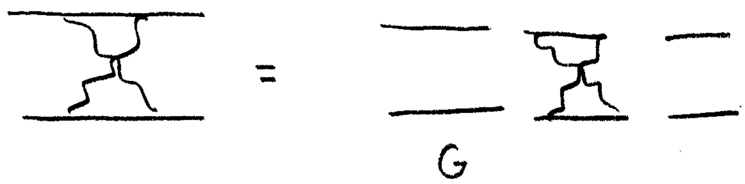
3.



a) G must be connected, up to terms like 

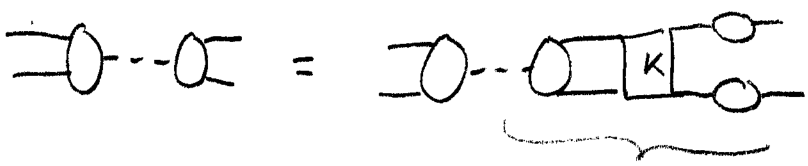
Eg.





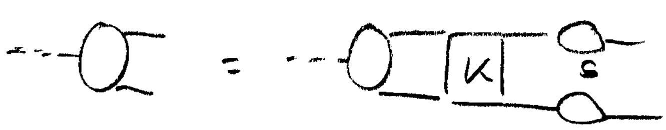
etc.

(b) Bound state pole implies

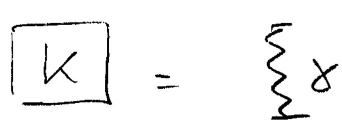


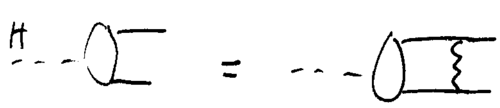
cannot have pole

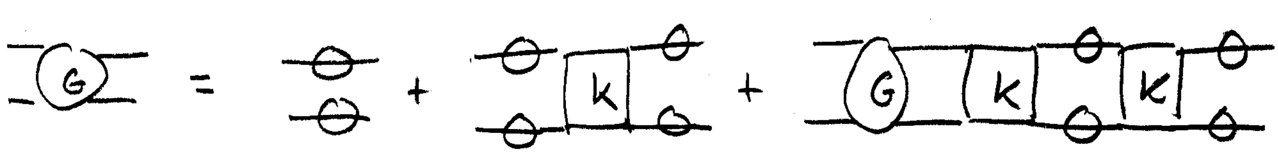
Factorize:



(c) H-atom







$\alpha = 1/137$ compensated by propagators almost on-shell.

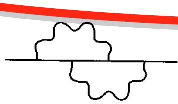
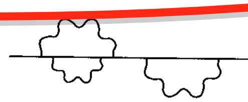
plus photon) state. In fact, it is a simple exercise in relativistic kinematics to show that the square root in (7.20), written in the form

$$k = \frac{1}{\sqrt{p^2}} \sqrt{\left[p^2 - (m_0 + \mu)^2\right]\left[p^2 - (m_0 - \mu)^2\right]},$$

is precisely the momentum in the center-of-mass frame for two particles of mass m_0 and μ and total energy $\sqrt{p^2}$. It is natural that this momentum becomes real at the two-particle threshold. The location of the branch cut is exactly where we would expect from the Källén-Lehmann spectral representation.†

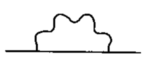
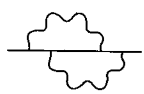
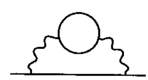
We have now located the two-particle branch cut predicted by the Källén-Lehmann representation, but we have not found the expected simple pole at $p^2 = m^2$. To find it we must actually include an infinite series of Feynman diagrams. Fortunately, this series will be easily summed.

Let us define a *one-particle irreducible* (1PI) diagram to be any diagram that cannot be split in two by removing a single line:

 is 1PI, while  is not.

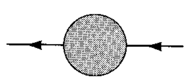
Let $-i\Sigma(p)$ denote the sum of all 1PI diagrams with two external fermion lines:

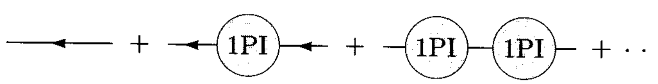
$-i\Sigma(p) =$ 

$=$  $+$  $+$  $+ \cdots$ (7.21)

(Although each diagram has two external lines, the Feynman propagators for these lines are not to be included in the expression for $\Sigma(p)$.) To leading order in α we see that $\Sigma = \Sigma_2$.

The Fourier transform of the two-point function can now be written as

$\int d^4x \, \langle\Omega|T\psi(x)\bar{\psi}(0)|\Omega\rangle \, e^{ip\cdot x} =$ 

$=$ 

$$= \frac{i(\not{p}+m_0)}{p^2-m_0^2} + \frac{i(\not{p}+m_0)}{p^2-m_0^2}(-i\Sigma)\frac{i(\not{p}+m_0)}{p^2-m_0^2} + \cdots. \tag{7.22}$$

†In real QED, $\mu = 0$ and the two-particle branch cut merges with the one-particle pole. This subtlety plays a role in the full treatment of the cancellation of infrared divergences, but it is beyond the scope of our present analysis.

for a diagram with E external lines. Eliminating I with the help of (6-69), we get $L - 1 + E/2 = V$ (E is even).

6-2-2 Truncated and Proper Diagrams

We introduce some terminology that will prove useful in the sequel.

The truncated functions are defined through the multiplication of Green functions in momentum space (without the δ^4 function of total energy momentum) by the inverse two-point functions pertaining to each external line:

$$G^{(n)}_{\text{trunc}}(p_1, \ldots, p_n) \equiv \prod_{k=1}^{n} [G^{(2)}(p_k, -p_k)]^{-1} G^{(n)}(p_1, \ldots, p_n) \qquad n > 2 \qquad (6\text{-}70)$$

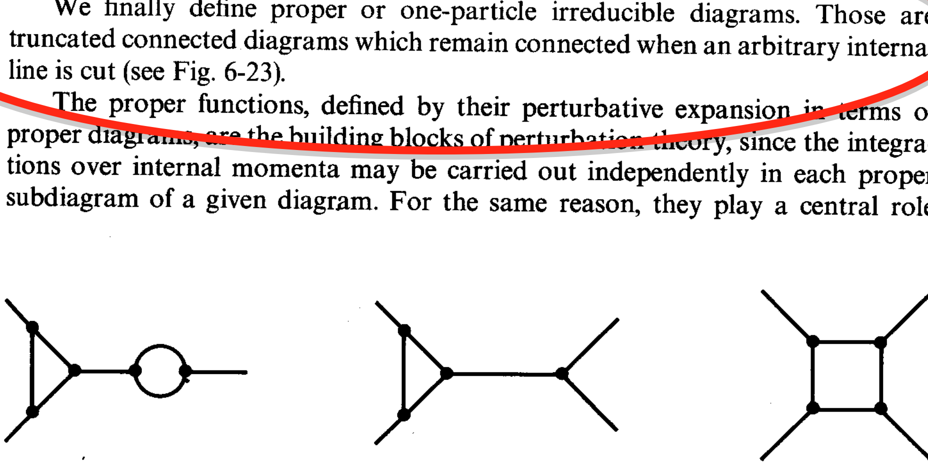
The two-point function $G^{(2)}$ is referred to as the complete propagator. For $p^2 \sim m^2$, we have $[G^{(2)}(p, -p)]^{-1} \sim (iZ)^{-1}(p^2 - m^2)$ where Z is the wave function renormalization introduced in Chap. 5. Hence, up to powers of Z, the on-shell values of these truncated functions are the quantities entering the reduction formulas. For instance, the connected part of the matrix element of Eq. (5-28) reads

$$\langle p_1, \ldots, p_n \text{ out} | q_1, \ldots, q_m \text{ in} \rangle_c = Z^{(n+m)/2} G^{(n+m)}_{\text{trunc}}(-p_1, \ldots, -p_n, q_1, \ldots, q_m) \Big|_{p_i^2 = q_j^2 = m^2} \times (2\pi)^4 \delta^4(\Sigma p_i - \Sigma q_j)$$

The perturbative expansion of truncated functions is expressed in terms of truncated diagrams, i.e., that have no self-energy part on their external lines. Moreover, in the Feynman rules, no factor or propagator is attributed to the external lines (see Fig. 6-23 for illustration). Finally, if we restore the factors $\hbar$, as indicated in the last subsection, L-loop truncated diagrams get a factor $\hbar^{L-1}$.

We finally define proper or one-particle irreducible diagrams. Those are truncated connected diagrams which remain connected when an arbitrary internal line is cut (see Fig. 6-23).

The proper functions, defined by their perturbative expansion in terms of proper diagrams, are the building blocks of perturbation theory, since the integrations over internal momenta may be carried out independently in each proper subdiagram of a given diagram. For the same reason, they play a central role



(a) (b) (c)

Figure 6-23 Examples of a nontruncated diagram (a), of a truncated but not proper diagram (b), and of a proper diagram (c). In cases (b) and (c), no factor is ascribed to the external lines.