1a) $q_\perp \to 0$: $\dfrac{1}{k_\perp^2 - 2x\,\vec{k}_\perp\cdot\vec{q}_\perp + x^2 q_\perp^2} \simeq \dfrac{1}{k_\perp^2} + 2x\,\dfrac{\vec{k}_\perp\cdot\vec{q}_\perp}{k_\perp^4}$

$$A = A_a + A_b \simeq -\frac{4e^3}{q_\perp^2}\,\frac{x(1-x)}{k_\perp^2}\,s\left[2\,\frac{\vec{k}_\perp\cdot\vec{q}_\perp}{k_\perp^2}\,\vec{e}_\lambda\cdot\vec{k}_\perp - \vec{e}_\lambda\cdot\vec{q}_\perp\right]$$

$$= \mathcal{O}\left(\frac{1}{q_\perp}\right) \quad \text{for } q_\perp \to 0$$

$\displaystyle\int \frac{d^2\vec{q}_\perp}{q_\perp^2}$: σ log divergent, but consider $m_\mu \neq 0$:

$$P = \left(\frac{m_\mu^2}{P^-}, P^-, \vec{0}_\perp\right); \qquad q^- \simeq \frac{M^2}{p^+} \quad \text{(as before)}$$

$$(P-q)^2 = m_\mu^2 - 2P\cdot q + q^2 = m_\mu^2 \;;$$

$$P^+q^- + P^-q^+ = \frac{m_\mu^2}{P^-}\,\frac{M^2}{p^+} + P^-q^+ = q^2 = q^+q^- - q_\perp^2$$

$$q^+ = \frac{-1}{P^-}\left[q_\perp^2 + \frac{m_\mu^2 M^2}{s}\right]$$

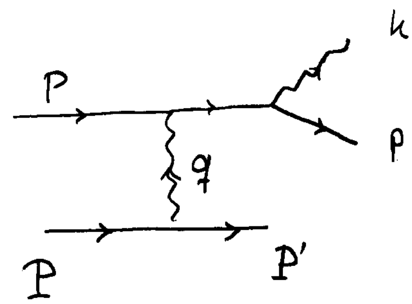
$$q^-q^+ = -\frac{M^2}{s}\left[q_\perp^2 + \frac{m_\mu^2 M^2}{s}\right]; \quad q^2 \simeq -\frac{m_\mu^2 M^4}{s^2} - q_\perp^2\left(1 + \frac{M^2}{s}\right)$$

$$\Rightarrow \int \frac{d^2\vec{q}_\perp}{q^4}\,q_\perp^2 \simeq \pi \int \frac{dq_\perp^2 \; q_\perp^2}{\left[q_\perp^2 + \frac{m_\mu^2 M^4}{s^2}\right]^2} \simeq \log\left(\frac{s^2}{m_\mu^2 M^2}\right)$$

$$p = (p^+, 0^-, \vec{0}_\perp)$$

$$P = \left(\frac{m_\mu^2}{P^-}, P^-, \vec{0}_\perp\right)$$

$$k = \left(x p^+, \frac{k_\perp^2}{x p^+}, \vec{k}_\perp\right)$$



1b) $\vec{k}_\perp \to 0$, $x\vec{q}_\perp$ with x, $\vec{q}_\perp$ fixed

$\vec{k}_\perp \to 0$: $A_a \sim \frac{1}{k_\perp}$; $\sigma \sim \int \frac{d^2 k_\perp}{k_\perp^2}$ log divergent

Collinear singularity. Consider $m_e \neq 0$:

$$p = \left(p^+, \frac{m_e^2}{p^+}, \vec{0}_\perp\right),$$

$$(p-k)^2 - m_e^2 = -2p\cdot k = -p^+k^- - p^-k^+ = -\frac{k_\perp^2}{x} - x m_e^2$$

$\Rightarrow$ Singularity regularized by m_e:

$$\int \frac{d^2\vec{k}_\perp \; k_\perp^2}{[k_\perp^2 + x^2 m_e^2]^2} \simeq \log\left(\frac{q_\perp^2}{m_e^2}\right)$$

$\vec{k}_\perp \to x\vec{q}_\perp$: Photon parallel to final electron. Compare angles:

$$\frac{\vec{k}_\perp}{x p^+} = \frac{\vec{q}_\perp - \vec{k}_\perp}{(1-x)p^+}; \qquad (1-x)\vec{k}_\perp = x(\vec{q}_\perp - \vec{k}_\perp)$$

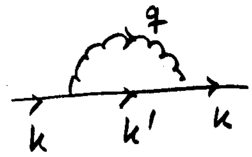
$$\vec{k}_\perp = x\vec{q}_\perp$$

In QED: Bremsstrahlung along incident & outgoing electron direction logarithmically enhanced. Related to emission of incoming photon cloud, and generation of outgoing photon cloud.

2. a) $iS(p) = i\,\dfrac{\not k + m}{p^2 - m^2 + i\varepsilon}$

$$i\,\mathrm{Im}\,S(p) = (\not k + m)(-i\pi)\,\delta(k^2 - m^2) = \tfrac{1}{2}\,\mathrm{Disc}\,S(p) \qquad \text{ok}$$

b) Use "truncated" expression



$k = k' + q$

$$i\Sigma(k) = (-ie\gamma^\mu)\int \frac{d^4k'}{(2\pi)^4}\, i\,\frac{\not k' + m}{k'^2 - m^2 + i\varepsilon}\,\frac{-i}{q^2 + i\varepsilon}\,(-ie)\gamma_\mu$$

$E_k' = \sqrt{\vec k'^2 + m^2}$

$$\mathrm{Im}\int \frac{d^4k'}{(2\pi)^4}\,\frac{i(\not k + m)}{(k'^0 - E_k' + i\varepsilon)(k'^0 + E_k' - i\varepsilon)}\,\frac{1}{(k^0 - k'^0 - |\vec k - \vec k'| + i\varepsilon)(k^0 - k'^0 + |\vec k - \vec k'| - i\varepsilon)}$$

(Close k'^0 in $\mathrm{Im}\,k'^0 < 0$ plane)

$$= \mathrm{Im}\left\{(-2\pi i)\, i\int \frac{d^3\vec k'}{(2\pi)^4}(\not k' + m)\left[\frac{\delta(k'^0 - E_k')}{2E_k'}\,\frac{1}{q^2 + i\varepsilon} + \frac{1}{k'^2 - m^2 + i\varepsilon}\,\frac{\delta(k^0 - k^{0\prime} + |\vec k - \vec k'|)}{2|\vec k - \vec k'|}\right]\right.$$

$$= \mathrm{Im}\left\{2\pi\int \frac{d^3\vec k'}{(2\pi)^4}(\not k' + m)\left[\frac{\delta(k'^0 - E_k)}{2E_k'}(-i\pi)\,\delta(q^2) + \frac{\delta(k^{0\prime} - k^0 - |\vec k - \vec k'|)}{2|\vec k - \vec k'|}(-i\pi)\,\delta(k'^2 - m^2)\right]\right.$$

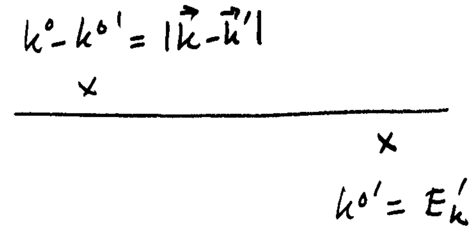
$$= \frac{1}{2}(-2\pi i)^2 \cdot \int \frac{d^4k'}{(2\pi)^4}(\not k' + m)\left[\Theta(k^{0\prime}) + \Theta(k^{0\prime} - k^0)\right]$$
$$* \,\delta(k'^2 - m^2)\,\delta(q^2)$$

$k^0 > 0 \Rightarrow k'^0 = k^0 - q^0 < k^0$ since physical photon has $q^0 > 0$
Only $\Theta(k^{0\prime})$ contributes

$k^0 < 0 \Rightarrow k'^0 < 0$ and only $\Theta(k^{0\prime} - k^0)$ contributes

2b) cont,

More precisely: Imaginary part arises from "pinch" of integ. contour, when poles at $+i\varepsilon$ and $-i\varepsilon$ occur at the same position



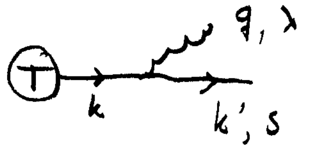
Eg: $\begin{cases} k^{0\prime} = E_k \\ k^0 = k^{0\prime} + |\vec{k}-\vec{k}'| \end{cases}$

$\Rightarrow k^0 = E_k + |\vec{k}-\vec{k}'|$

(i.e. $k^{0\prime} > 0$ and $q^0 > 0$ are coupled)

Since only one of the θ-functions can contribute:

Disc is obtained by replacement of the two denominators by $(-2\pi i)^2 \delta(k'^2 - m^2)\, \delta(q^2)$

3. $\Big|$  $\Big|^2$

$$\sigma \sim \sum_{s,\lambda} \int \frac{d^3\vec{k}'\, d^3\vec{q}}{(2\pi)^6\, 2E_k'\, 2|\vec{q}|} \left| \bar{u}(k',s)\left[-ie\, \not{\varepsilon}^*(q,\lambda)\right] i\, \frac{\not{k}+m}{k^2-m^2}\, T \right|^2$$

$$\sum_\lambda \varepsilon^*_\mu(q,\lambda)\, \varepsilon_\nu(q,\lambda) \to -g_{\mu\nu}$$

$$\sum_s u(k',s)\,\bar{u}(k',s) = \not{k}' + m \qquad\qquad \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$$

$$\int \frac{d^3\vec{k}'}{(2\pi)^3\, 2E_k'} = \int \frac{d^4k'}{(2\pi)^4}\, \Theta(k^{0\prime})\, 2\pi\, \delta(k'^2-m^2)$$

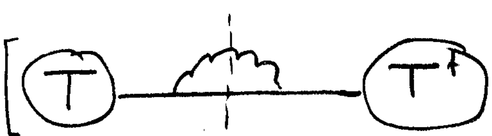
$$\sigma \sim \int \frac{d^4k'\, d^4q}{(2\pi)^8}\, \Theta(k^{0\prime})\, \Theta(q^0)\, 2\pi\, \delta(k'^2-m^2)\, 2\pi\, \delta(q^2)$$

$$* \sum_s T^\dagger \gamma^0\, \frac{\not{k}+m}{k^2-m^2}\, e\gamma_\mu\, u(k',s)\,\bar{u}(k',s)\, e\gamma_\nu\, \frac{\not{k}+m}{k^2-m^2}\, T\, (-g^{\mu\nu})$$

$$\int d^4q \to \int d^4k \qquad\qquad (k = k'+q)$$

$$\sigma \sim \int \frac{d^4k}{(2\pi)^4}\, \Theta(k^0) \int \frac{d^4k'}{(2\pi)^4}\, T^\dagger\gamma^0\, i\, \frac{\not{k}+m}{k^2-m^2}\, (-ie\gamma_\mu)$$

$$* \; i(\not{k}'+m)(-2\pi i)\,\delta(k'^2-m^2)\,(-ie\gamma_\nu)\, i\, \frac{\not{k}+m}{k^2-m^2}\, T\, (-ig^{\mu\nu})(-2\pi i)\,\delta(q^2)$$

$$= i \cdot \mathrm{Disc}\left[\right.$$  $$\left.\right]$$