Session of Monday 6 October at 16.00-17.30 in aud A315.

1. (a) Prove the following identity for a fermion propagator:

$$\frac{\not{p}+m}{p^2-m^2+i\varepsilon} = \frac{1}{2E_p}\sum_s \left[\frac{u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)}{p^0-E_p+i\varepsilon} + \frac{v(-\boldsymbol{p},s)\bar{v}(-\boldsymbol{p},s)}{p^0+E_p-i\varepsilon}\right]$$

   where $E_p = +\sqrt{\boldsymbol{p}^2+m^2}$. Note that the spinors are independent of $p^0$.
   Do a Fourier transform of the propagator from $p^0$ to time $t$, and give an interpretation of the two terms on the *rhs*.

   (b) Find the corresponding identity in terms of the LF spinors defined in eqs. (14) - (15) of the "LF Spinors and Polarization Vectors" file on the home page, where the '$-$' momentum in the spinors is defined to be $p^-_{LF} \equiv (\boldsymbol{p}_\perp^2+m^2)/p^+$.
   Do a Fourier transform from $p^-$ to $x^+$ and give your interpretation.

2. (a) Calculate the amplitude for a massless electron of momentum $p = (p^+, 0^-, \boldsymbol{0}_\perp)$ and positive helicity to radiate a photon with momentum $k = (xp^+, k_\perp^2/xp^+, \boldsymbol{k}_\perp)$ and helicity $\lambda = \pm 1$. You may use the LF spinors and polarization vectors given on the home page.

   (b) Discuss the dependence of the photon polarization on $x$. Find the relation between the squared amplitude and the splitting function $P_{\gamma/e}(x) = [1+(1-x)^2]/x$.

3. Protons may emit soft photons coherently, *i.e.*, photons with a wavelength longer than the proton radius, which couple to the total charge of the proton.

   (a) Estimate the range in transverse momentum and energy of photons coherently emitted by the 7 TeV protons in the LHC. Are they interesting for the LHC experiments?

   (b) Evaluate the off-shellness $p^2 - m^2$ of the emitting proton in the process $N(p) \to N(p') + \gamma(k)$. For a given $x = k^+/p^+$, estimate the range of $\boldsymbol{k}_\perp$ in which the off-shellness is independent of the transverse momentum of the photon. From this, determine the effective radius of the photon cloud as a function of $x$. What is this radius (in fm) at the LHC for photons of 100 GeV?

1 a) $p^2 - m^2 + i\varepsilon = (p^0 - E_p + i\varepsilon)(p^0 + E_p - i\varepsilon)$

Check the common numerator on the rhs:

$$\frac{1}{2E_p}\left[(p^0+E_p)(E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m) + (p^0 - E_p)(E_p\gamma^0 + \vec{p}\cdot\vec{\gamma} - m)\right]$$

$$= p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} + m = \text{lhs.}$$

$$S(t,\vec{p}) = \int \frac{dp^0}{2\pi} S(p^0,\vec{p})\, e^{-itp^0} =$$

$$= \left[\Theta(t)\int_{\circlearrowright} \frac{dp^0}{2\pi} + \Theta(-t)\int_{\circlearrowleft} \frac{dp^0}{2\pi}\right] S(p^0,\vec{p})\, e^{-itp^0}$$

$$= -\frac{i}{2E_p}\sum_s \Big[\Theta(t)\, u(\vec{p},s)\bar{u}(\vec{p},s)\, e^{-itE_p}$$
$$- \Theta(-t)\, v(-\vec{p},s)\bar{v}(-\vec{p},s)\, e^{itE_p}\Big]$$

$\Rightarrow$ First term is fermion propagating forward in $t$

Second — " — anti — " — backward — " —

b) LF spinors: $p^- \equiv \dfrac{p_\perp^2 + m^2}{p^+}$

$$\sum_s u(p,s)\bar{u}(p,s) = \frac{1}{p^+}\sum_s (\not{p}+m)\underbrace{\tilde{\chi}(s)\chi^\dagger(s)}_{U_{s,s} = \frac{1}{4}\not{n}(1+s\gamma_5)}(\not{p}+m)$$

$$= \frac{1}{2p^+}(\not{p}+m)\not{n}(\not{p}+m)$$

$$= \frac{1}{2p^+}(\not{p}+m)\big[(-\not{p}+m) + 2\overbrace{n\cdot p}^{p^+}\big] = \not{p}+m$$

1b cont.

$$\sum_s v(p,s)\bar{v}(p,s) = \frac{1}{2p^+}(\not{p}-m)\not{n}(\not{p}-m) = \not{p}-m$$

$\Rightarrow$ As for ordinary spinors, but $p^- = \dfrac{p_\perp^2+m^2}{p^+} \equiv p_{LF}^-$

$$\sum_s u(p,s)\bar{u}(p,s) = \not{p}_{LF}+m = \tfrac{1}{2}p^+\gamma^- + \tfrac{1}{2}p_{LF}^-\gamma^+ - \vec{p}_\perp\cdot\vec{\gamma}_\perp + m$$

$$= \tfrac{1}{2}p^+\gamma^- + \tfrac{1}{2}p^-\gamma^+ - \vec{p}_\perp\cdot\vec{\gamma}_\perp + m + \tfrac{1}{2}(p^- - p_{LF}^-)\gamma^+$$

$$p^2-m^2+i\varepsilon = p^+p^- - p_\perp^2 - m^2 + i\varepsilon = p^+(p^- - p_{LF}^-) + i\varepsilon$$

$$\frac{\not{p}+m}{p^2-m^2+i\varepsilon} = \sum_s \frac{u(p,s)\bar{u}(p,s)}{p^+(p^- - p_{LF}^-)+i\varepsilon} + \frac{1}{2p^+}\gamma^+$$

$$\int \frac{dp^-}{2\pi} S(p^+,p^-,\vec{p}_\perp)e^{-\frac{i}{2}x^+p^-} = -i\,\Theta(x^+)\Theta(p^+)\frac{1}{p^+}\underbrace{\sum_s u(p,s)\bar{u}(p,s)}e^{-ix^+p_{LF}^-/2}$$

$$+\, i\cdot\Theta(-x^+)\Theta(-p^+)\frac{1}{p^+}\sum_s u(p,s)\bar{u}(p,s)e^{-ix^+p_{LF}^-/2}$$

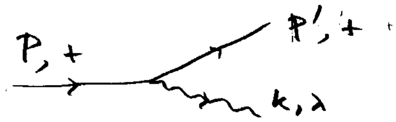
$$+\,\delta(x^+/2)\gamma^+/2p^+$$

$$= -\frac{i}{|p^+|}\Theta(x^+p^+)\sum_s u(p,s)\bar{u}(p,s)e^{-ix^+p_{LF}^-/2} + \frac{\gamma^+}{p^+}\delta(x^+)$$

* Fermions with $p^+ \gtrless 0$ propagate $\begin{matrix}\text{forward}\\ \text{backward}\end{matrix}$ in $x^+$

* There is an "instantaneous" (in $x^+$) component, which may be thought of as the backward-propagating (in $t$) anti-fermion boosted to the $\infty$ momentum frame.

2a) Helicity is conserved for $m=0$ fermions




$$p = (p^+, 0^-, \vec{0}_\perp) = \tfrac{1}{2} p^+ \tilde{n}$$

$$iA_\lambda = \bar{u}(p', +)(-ie\, \not{\varepsilon}^*_\lambda(k))\, u(p,+)$$

$$u(p,+) = \frac{1}{\sqrt{p^+}} \not{p}\, \tilde{\chi}(+)$$

Note : $p'^- = \dfrac{p'^{\,2}_\perp}{p'^+}$ in LF spinor $\neq p^- - k^-$

$$A_\lambda = -\frac{e}{\sqrt{p^+ p'^+}} \mathrm{Tr}\left[\not{p}' \not{\varepsilon}^*_\lambda(k) \not{p}\, \underbrace{\tilde{\chi}(+)\chi^+(+)}_{U_{++} = \frac{1}{4}\not{n}(1+\gamma_5)}\right]$$

$$\begin{pmatrix} n\cdot p = p^+ \\ n\cdot n = 0 \end{pmatrix}$$

$$= -\frac{e}{4\sqrt{p^+ p'^+}} \mathrm{Tr}\left[\not{p}' \not{\varepsilon}^*_\lambda \not{p}\, \not{n}(1+\gamma_5)\right] \qquad (\lambda = \pm 1)$$

$$\varepsilon_\lambda = e_\lambda - \frac{e_\lambda \cdot k}{k^+} n \quad ; \quad e_\lambda = (0, 0, \vec{e}_\lambda) \; ; \; \vec{e}_\lambda = -\frac{1}{\sqrt{2}}(\lambda, i)$$

$(\lambda = \pm 1)$

$$e_\lambda \cdot k = \lambda \frac{k_\perp}{\sqrt{2}} e^{i\lambda\varphi} \; ; \quad \vec{k}_\perp = k_\perp(\cos\varphi, \sin\varphi)$$

$$n \cdot \varepsilon^*_\lambda = 0$$

$$p \cdot \varepsilon^*_\lambda = -\frac{\lambda}{\sqrt{2}\, x} k_\perp e^{-i\lambda\varphi}$$

$$p' \cdot \varepsilon^*_\lambda = -k \cdot \overset{\circ}{e}{}^*_\lambda - \frac{\overset{\circ}{e}{}^*_\lambda \cdot k}{x p^+}(1-x) p^+ = -\frac{e^*_\lambda \cdot k}{x} = p \cdot \varepsilon^*_\lambda$$

$$A_\lambda = -\frac{e}{p^+\sqrt{1-x}} \left\{ \left[ p^+(p' \cdot \varepsilon^*_\lambda) + p'^+(p\cdot\varepsilon^*_\lambda)\right] + \frac{p^+}{8} \mathrm{Tr}\left[\not{p}' \not{\varepsilon}^*_\lambda \tilde{\not{n}} \not{n} \gamma_5\right] \right\}$$

Use identity: $\not{n}\gamma_5 \not{e}_\lambda = \lambda \not{n} \not{e}_\lambda$ (8)

$$e^*_\lambda = -e_{-\lambda}$$

$p' \to -k$ in trace (only $\perp$ contrib.)

2a cont.)

$$\mathrm{Tr} = -\mathrm{Tr}\left[ \not{k}\, \tilde{\not{n}}\, \not{n}\, \gamma_5\, \not{e}_{-\lambda} \right] = +\lambda\, \mathrm{Tr}\left[ \not{k}\, \tilde{\not{n}}\, \not{n}\, \not{e}_{-\lambda} \right] \qquad \Big|\ n \cdot \tilde{n} = 2$$

$$= +8\lambda\, k \cdot e_{-\lambda} = -\frac{8}{\sqrt{2}} k_\perp e^{-i\lambda\varphi} \qquad (\lambda^2 = 1)$$

$$A_\lambda = +\frac{e}{\sqrt{1-x}} \frac{k_\perp e^{-i\lambda\varphi}}{\sqrt{2}\, x} \left[ \lambda + (1-x)\lambda + x = (1+\lambda) - (1-\lambda)(1-x) \right]$$

b)

$$\frac{A_{-1}}{A_{+1}} = -(1-x) \quad \begin{cases} \to 0 \text{ for } x \to 1: & \gamma \text{ carries } e \text{ spin direction} \\ \to -1 \text{ for } x \to 0: & \gamma \text{ unpolarized} \end{cases}$$

$$\sum_\lambda |A_\lambda|^2 = \frac{2e^2 k_\perp^2}{x(1-x)} \frac{1+(1-x)^2}{x} = \frac{2e^2 k_\perp^2}{x(1-x)} P_{e\to\gamma}(x)$$

3. $p \to p + \gamma$ coherently

$$\Rightarrow \; p_\gamma \lesssim \frac{1}{r_p} \simeq \frac{1}{1\ \text{fm}} \simeq 200\ \text{MeV}$$

a) At the LHC: $\quad p_\perp \lesssim 200$ MeV

$$p_\parallel \lesssim 0.2\ \text{GeV} \cdot \frac{E_p}{m_p} = 0.2 \cdot \frac{7 \cdot 10^3}{.94} = 1.5\ \text{GeV}$$

$$x = \frac{E_\gamma}{E_p} \simeq \frac{1}{r_p}\,\frac{E_p}{m_p}\,\frac{1}{E_p} = \frac{1}{m_p r_p} \simeq 0.2$$

Yes: 1.5 GeV $\gamma$ + 7 TeV $p$ gives

$$E_{CM} \simeq \sqrt{4 \cdot 1.5 \cdot 7} = 6.5\ \text{TeV} \text{ is interesting!}$$

b) $$(p'+k)^2 - m_p^2 = 2p' \cdot k = p'^+ k^- + p'^- k^+ + 2k_\perp^2$$

$$= (1-x)p^+ \frac{k_\perp^2}{x p^+} + \frac{k_\perp^2 + m_p^2}{(1-x)p^+}\, x p^+ + 2k_\perp^2 =$$

$$= \frac{x}{1-x} m_p^2 + k_\perp^2 \left[ \frac{1-x}{x} + \frac{x}{1-x} + 2 = \frac{(1-x)^2 + x^2 + 2x(1-x)}{x(1-x)} = \frac{1}{x(1-x)} \right]$$

$$= \frac{1}{x(1-x)} \left[ k_\perp^2 + (x m_p)^2 \right]$$ which is insensitive to $k_\perp$ for

$$k_\perp \lesssim x m_p \;\Rightarrow\; r_\gamma \simeq \frac{1}{k_\perp} = \frac{1}{x m_p}$$

$$x = \frac{0.1}{7} \Rightarrow r_\gamma = \frac{7}{0.1 \cdot 0.94\ \text{GeV}}\, 0.2\ \text{fm GeV} = 15\ \text{fm}$$

Note: $r_\gamma \to \infty$ for $x \to 0$: EM has $\infty$ range