QCD and Hadron Structure Fall 2008 **Exercise 6**

Session of Monday 13 October at 16.00-17.30 in aud A315.

1. Derive the expression for the *equivalent photon approximation* (also known as the *Weizsäcker-Williams approximation*) of photon emission.

   (a) Using the results of Ex. 5, problems 2 and 3b, write the square of the amplitude for a virtual electron of momentum $p = (p^+, p^2/p^+, \mathbf{0})$ to emit a photon with $k = (xp^+, \boldsymbol{k}^2/k^+, \boldsymbol{k}_\perp)$. Assume $p^+ \gg \sqrt{p^2}$ so that you may keep only $p^- = p^-_{LF}$ pole part of the electron propagator. You should sum over the photon polarization and keep the electron mass m only in the virtual electron propagator.

   (b) Write the differential emission probability in x and $k_\perp$ by including the phase space factor.

   (c) Assuming that the virtual electron emerges from a process with hardness scale Q, integrate over $k_\perp$ to find the differential emission probability as a function of x:

$$P(e \to e\gamma) = \frac{\alpha}{2\pi} \log\left[\frac{Q^2}{(xm)^2}\right] \int dx \, \frac{1+(1-x)^2}{x} \tag{1}$$

2. Let the momentum distribution of the electron from the hard process of 1(c) be $f_e(z, Q^2)$, where $z = p^+/P^+$ is the momentum fraction of some initial momentum P.

   (a) Based on the result (1), show that the Q^2-dependence of the electron momentum distribution is

$$\frac{df_e(z,Q^2)}{d\log Q^2} = \frac{\alpha}{2\pi}\int_z^1 \frac{dy}{y}\left[\frac{1+y^2}{(1-y)_+} + A\delta(1-y)\right] f_e(\frac{z}{y}, Q^2) \tag{2}$$

   Here the splitting function has been regularized by the '+' prescription: $\int dx f(x)/(1-x)_+ \equiv \int dx [f(x) - f(1)]/(1-x)$ and a δ-function contribution was added to account for the the virtual photon loop contribution.

   (b) Determine A in (2) from the electron number constraint, $\int dz f_e(z, Q^2) = 1$.

3. Apply the above result to $q \to q + g$ splitting in QCD.

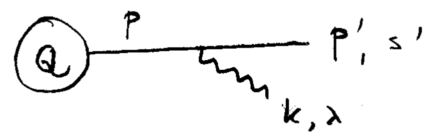
   (a) Evaluate the color factor of this process, by summing over the final and averaging over the initial colors.

   (b) Determine the Q^2-dependence of the moments $M_n(Q^2) \equiv \int_0^1 dz \, z^{n-1} f_q(z, Q^2)$.

   (c) Show that $f_q(z, Q^2)$ decreases (increases) with Q^2 at high (low) z.

1.

a)



$$iM_\lambda = \bar{u}(p',s')(-ie\,\not{\varepsilon}^*_\lambda(k))\,u(p,s)\,iT(Q,p,s)$$

$$* \frac{i}{p^+(p^- - p^-_{LF}) + i\varepsilon}$$

Note: $\bar{u}(p,s)$ included in T
$s = s'$ $(+\mathcal{O}(m))$

$$p^+(p^- - p^-_{LF}) = p^2 - m^2 = \frac{1}{x(1-x)}\left[k_\perp^2 + (xm)^2\right] \quad \left(\begin{array}{c}5/08\\ \text{Exc. } 2\end{array}\right)$$

$$\sum_\lambda |e\,\bar{u}(p',s')\,\not{\varepsilon}^*_\lambda(k)\,u(p,s)\,T|^2 = \frac{2e^2 k_\perp^2}{x(1-x)}\,\frac{1+(1-x)^2}{x}\,|T|^2$$

$$\sum_\lambda |M_\lambda|^2 = \frac{2e^2 k_\perp^2}{x(1-x)}\,\frac{1+(1-x)^2}{x}\,|T|^2\,\frac{x^2(1-x)^2}{[k_\perp^2+(xm)^2]^2}$$

$$= \frac{2e^2 k_\perp^2(1-x)}{[k_\perp^2+(xm)^2]^2}\left[1+(1-x)^2\right]|T|^2$$

b)
$$\int \frac{d^3\vec{p}\,'\,d^3\vec{k}}{(2\pi)^6\,2E_p'\,2|\vec{k}|} = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\,\frac{d^3\vec{k}}{(2\pi)^3 2|\vec{k}|}\,\frac{E_p}{E_p'} \quad \left|\quad \begin{array}{l} |\vec{k}| \simeq k^z \\ E_{p'} \simeq (1-x)E_p \end{array}\right.$$

$$\simeq \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\,\frac{dk^z\,k_\perp dk_\perp d\varphi}{(2\pi)^3\,2\,k^z(1-x)} = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}\,\frac{dx}{x}\,\frac{dk_\perp^2}{(2\pi)^2\cdot 4(1-x)}$$

Now $\displaystyle\int \frac{d^3\vec{p}}{(2\pi)^3 2E_k}\,|T|^2$ is emission prob. of electron $\Rightarrow$

$$P(e\to e\gamma) = \int \frac{dx}{x}\,\frac{e^2}{8\pi^2}\,\frac{dk_\perp^2\,k_\perp^2}{[k_\perp^2+(xm)^2]^2}\left[1+(1-x)^2\right]$$

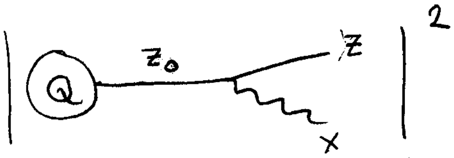
$$= \frac{\alpha}{2\pi}\int dx\,dk_\perp^2\,\frac{k_\perp^2}{[k_\perp^2+(xm)^2]^2}\,\frac{1+(1-x)^2}{x}$$

1c) If $|T|^2$ is insensitive to p^2 for $p^2 \lesssim Q^2$ we may integrate $k_\perp^2 \lesssim Q^2$

$$P(e \to e\gamma) \simeq \frac{\alpha}{2\pi} \log\left(\frac{Q^2}{x^2 m^2}\right) \int \frac{1+(1-x)^2}{x} dx$$

In the equiv. photon approx $\log\left(\frac{Q^2}{x^2m^2}\right) \to \log\left(\frac{Q^2}{m^2}\right)$, which is ok if $\frac{1}{x} \lesssim \frac{Q}{m}$.

(The $\log x$ behaviour for $x \to 0$ is related to the soft photon IR singularity, whereas $\log(Q/m)$ comes from the collinear singularity).

2a) The Q^2-dep. arising from the splitting:  $y \equiv 1-x$

$$\frac{d}{d\log Q^2} f_e(z,Q^2) = \int dz_0\, dx\, \frac{\alpha}{2\pi} \frac{1+(1-x)^2}{x} f_e(z_0,Q^2)\,\delta(z - z_0(1-x))$$

$$= \frac{\alpha}{2\pi}\int_z^1 \frac{dy}{y} \frac{1+y^2}{1-y} f_e\left(\frac{z}{y}\right)$$

However, at $\mathcal{O}(\alpha)$ we also have  + h.c.

which contributes at $y=1$ $(x=0)$

It is easiest to determine the size of this term from the conservation of electron number.

First regularize the integrand at $y=1$, then add δ-fn:

$$\frac{d}{d\log Q^2} f_e(z,Q^2) = \frac{\alpha}{2\pi}\int_z^1 \frac{dy}{y}\left[\frac{1+y^2}{(1-y)_+} + A\,\delta(1-y)\right] f_e\left(\frac{z}{y}\right)$$

Note: $\frac{1+y^2}{(1-y)_+} = \frac{1+y^2-2}{1-y} = -(1+y)$ Taking $f_e(z/y)_{y=1} = f_e(z)$ into '+' def. only changes A.

2b) $\int dz\, f_e(z,Q^2) = 1$ => LHS must vanish after $\int_0^1 dz$

$$\int_0^1 dz \int_z^1 \frac{dy}{y} = \int_0^1 dy \int_0^y \frac{dz}{y} = \int_0^1 dy \int_0^1 du\,, \quad (u = z/y)$$

$$\int_0^1 dy\, du\left[-(1+y) + A\,\delta(1-y)\right] f_e(u) = \int_0^1 dy\left[-(1+y) + A\,\delta(1-y)\right] = 0$$

$\Rightarrow A = 3/2$

to lower x by radiation, so that the integral over the full term of order α is zero. Another way of expressing this criterion is that A is determined by the condition that the electron contain exactly one electron parton,

$$\int_0^1 dx\, f_e(x) = 1. \tag{17.103}$$

(This equation will be modified below, when we include pair-creation processes.)

It is not so clear how to integrate over the singular denominator in (17.100) to determine A explicitly. It is conventional to define a distribution that can be integrated by subtracting a delta function from the singular term. Define the distribution

$$\frac{1}{(1-x)_+} \tag{17.104}$$

to agree with the function $1/(1-x)$ for all values of x less than 1, and to have a singularity at $x = 1$ such that the integral of this distribution with any smooth function $f(x)$ gives

$$\int_0^1 dx\, \frac{f(x)}{(1-x)_+} = \int_0^1 dx\, \frac{f(x) - f(1)}{(1-x)}. \tag{17.105}$$

Less formally,

$$\frac{1}{(1-x)_+} = \lim_{\epsilon\to 0}\left[\frac{1}{(1-x)}\theta(1-x-\epsilon) - \delta(1-x)\int_0^{1-\epsilon} dx' \frac{1}{(1-x')}\right]. \tag{17.106}$$

The more formal definition (17.105) is often easier to use in practice.

Using this definition, we can bring a piece of the delta function into the singular term of (17.102) by changing the denominator $(1-x)$ to $(1-x)_+$. Then, to normalize (17.102), we need the integral

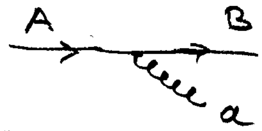
$$\int_0^1 dx\, \frac{1+x^2}{(1-x)_+} = \int_0^1 dx\, \frac{x^2-1}{(1-x)} = -\frac{3}{2}.$$

Our final form of the electron distribution, to order α, is

$$f_e(x) = \delta(1-x) + \frac{\alpha}{2\pi}\log\frac{s}{m^2}\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]. \tag{17.107}$$

This distribution is now properly normalized, but it is still highly singular near $x = 1$. Thus, we should expect higher-order corrections to the electron distribution function to be important in this region. We must, then, think about how to treat the emission of many collinear photons.

3a) Color factor



$$C = \frac{1}{N_c} \sum_{\substack{A,B \\ a}} T^a_{BA} T^a_{AB}$$

$$\left| \begin{array}{l} T^a T^a = C_F \mathbb{1} = \frac{N_c^2 - 1}{2N_c} \mathbb{1} \\ \mathrm{Tr}\, \mathbb{1} = N_c \end{array} \right.$$

$$= \frac{1}{N_c} \mathrm{Tr}\left(\sum_a T^a T^a \right) =$$

$$= C_F = \frac{4}{3} \text{ for } N_c = 3$$

3b)

$$u = \frac{z}{y} \; ; \; dz = y\, du$$

$$\frac{dM_n}{d\log Q^2} = \frac{\alpha_s C_F}{2\pi} \int_0^1 dz\, z^{n-1} \int_z^1 \frac{dy}{y} \left[\frac{1+y^2}{(1-y)_+} + \frac{3}{2}\delta(1-y) \right] f_q\left(\frac{z}{y}, Q^2\right)$$

$$= \frac{\alpha_s C_F}{2\pi} \int_0^1 du\, u^{n-1} f_q(u, Q^2) \int_0^1 dy\, y^{n-1} \left[\frac{1+y^2}{(1-y)_+} + \frac{3}{2}\delta(1-y) \right]$$

$$= \frac{\alpha_s C_F}{2\pi} M_n(Q^2)\, a_n$$

$$a_n - \frac{3}{2} = \int_0^1 dy \left| \frac{y^{n+1} + y^{n-1} - 2}{(1-y)} \right.$$

Let $S_n = \sum_{k=0}^{n} y^k \Rightarrow y S_n = S_n - 1 + y^{n+1}$

$$S_n = \frac{1 - y^{n+1}}{1-y}$$

$$a_n = \frac{3}{2} - \int_0^1 dy\, (S_n + S_{n-2}) = \frac{3}{2} - \int dy\, (2S_{n-2} + y^{n-1} + y^n)$$

$$= \frac{3}{2} - \int_0^1 dy \left[2\sum_{k=0}^{n-2} y^k + y^{n-1} + y^n \right]$$

3b cont.)

$$a_n = \frac{3}{2} - 2\sum_{k=0}^{n-2} \frac{1}{k+1} - \frac{1}{n} - \frac{1}{n+1} \qquad \propto \text{"anomalous dimensions"}$$

$$d\log M_n = \frac{\alpha_s C_F}{2\pi} a_n \, d\log Q^2$$

$$\alpha_s(Q^2) = \frac{4\pi}{b_0 \log(Q^2/\Lambda^2)} \quad ; \quad b_0 = 11 - \frac{2}{3} n_f$$

$$d\log M_n = \frac{2 C_F a_n}{b_0} \, d\log\log Q^2/\Lambda^2$$

$$M_n(Q^2) = \left[\frac{\log(Q^2/\Lambda^2)}{\log(\mu^2/\Lambda^2)}\right]^{\frac{2 C_F a_n}{b_0}} \qquad \log Q^2\text{- dependence}$$

3c) $a_1 = 0$ (conservation of quark number)

$a_2 < 0$ (loss of energy to gluons)

$a_n < 0$ for $n \geq 2$

$\Rightarrow$ $M_n(Q^2)$ decreases with Q^2 for $n \geq 2$

But for $n \geq 2$ the high z part of $f_q(z, Q^2)$ is weighted

Since $\int dz \, f_q(z)$ is indep. of Q^2,

$$f_q(z) \begin{cases} \text{decreases} \\ \text{increases} \end{cases} \text{at} \begin{matrix} \text{high} \\ \text{low} \end{matrix} \; z$$