Session of Monday 20 October at 16.00-17.30 in aud A315.

1. The elastic form factors $F_{1,2}(Q^2)$ of the proton are defined in terms of the photon current matrix elements $F^\mu_{\lambda,\lambda'}$ as

$$F^\mu_{\lambda,\lambda'} \equiv \langle p+q/2, \lambda'|J^\mu(0)|p-q/2,\lambda\rangle = \bar{u}(p+q/2,\lambda')\left[F_1(Q^2)\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(Q^2)\right]u(p-q/2,\lambda)$$

where $Q^2 = -q^2$, λ, λ' are the proton helicities and m is the proton mass. The photon current (for a given quark q) is $J^\mu(x) = \bar{\mathrm{q}}(x)\gamma^\mu \mathrm{q}(x)$, where $\mathrm{q}(x)$ is the quark field operator.

   (a) Show that $J^+(x) = \mathrm{q}_+^\dagger(x)\mathrm{q}_+(x)$ and is thus a density operator of the quark field $\mathrm{q}_+(x) \equiv \gamma^-\gamma^+\mathrm{q}(x)/4$, as stated in the lecture of Vanderhaeghen I.41.

   (b) Evaluate $F^+_{\lambda,\lambda'}$ in the frame where $q^+ \equiv q^0 + q^3 = 0$ and $\boldsymbol{p}_\perp = 0$, using the LF spinors on the home page. Verify that only $F_1(Q^2)$ contributes to $F^+_{\frac{1}{2},\frac{1}{2}}$.

   (c) Express the states $|p \pm q/2, s_\perp = \frac{1}{2}\rangle$, where the proton spin is orthogonal to to its momentum, in terms of the helicity states, and evaluate $\langle p+q/2, s_\perp = \frac{1}{2}|J^\mu(0)|p-q/2, s_\perp = \frac{1}{2}\rangle$. Compare with the expression for the transverse quark density on Vdh I.42.

2. The Generalized Parton Distributions H and E are defined by Vdh II.4 as

$$\frac{p^+}{2\pi}\int_{-\infty}^{\infty} dy^- e^{ixp^+y^-}\langle p+\tfrac{1}{2}\Delta, \lambda'|\bar{\mathrm{q}}(-\tfrac{1}{2}y)\gamma^+\mathrm{q}(\tfrac{1}{2}y)|p-\tfrac{1}{2}\Delta,\lambda\rangle_{y^+=y_\perp=0}$$

$$= \bar{u}(p+\tfrac{1}{2}\Delta,\lambda')\left[H(x,\xi,t)\gamma^+ + E(x,\xi,t)\frac{i\sigma^{+,\nu}\Delta_\nu}{2m}\right]u(p-\tfrac{1}{2}\Delta,\lambda)$$

where $\Delta^+ = -2\xi p^+$ and $t = \Delta^2$. Prove the relations on Vdh II.5:

$$\int_{-1}^{1} dx\, H(x,\xi,t) = F_1(t) \qquad \text{and} \qquad \int_{-1}^{1} dx\, E(x,\xi,t) = F_2(t)$$

Hint: The matrix element on the *lhs.*, as well as H and E, vanish for $|x| > 1$.

3. The Single Spin Asymmetry (SSA) of a scattering cross section $\sigma(s_a) \equiv \sigma(\vec{a} + b \to c + d)$ is defined as $A_{s_a} \equiv [\sigma(\vec{s}_a) - \sigma(-\vec{s}_a)]/[\sigma(\vec{s}_a) + \sigma(-\vec{s}_a)]$. Here $\vec{a}$ is a $j = \frac{1}{2}$ particle polarized in the direction $\vec{s}_a$. The spins of particles b, c and d are summed over. Particle a moves along the positive z-axis and the scattering angle $\theta_{ac} \neq 0, \pi$.

   (a) Let $T(\lambda_a, \lambda_b, \lambda_c, \lambda_d)$ be the scattering amplitude in the helicity basis ($\lambda_a = \pm\frac{1}{2}$ for $\vec{s}_a = \pm\hat{z} = (0,0,\pm 1)$). Show that parity invariance implies $\sigma(\lambda_a = \frac{1}{2}) = \sigma(\lambda_a = -\frac{1}{2})$, *i.e.*, $A_z = 0$. (*Hint:* Helicity transforms as you would expect classically.)

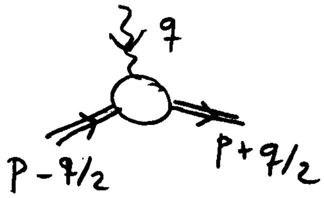
   (b) Derive the expression for the asymmetry when $\vec{s}_a = \hat{y} = (0,1,0)$,

$$A_y = \frac{2\,\mathrm{Im}\sum_{\lambda_b,\lambda_c,\lambda_d} T_+T_-^*}{\sum_{\lambda_b,\lambda_c,\lambda_d}(|T_+|^2 + |T_-|^2)} \qquad \text{where} \qquad T_\pm \equiv T(\lambda_a = \pm\tfrac{1}{2}, \lambda_b, \lambda_c, \lambda_d).$$

   Thus $A_y \neq 0$ requires both spin flip and a dynamical phase.

1. $F^{\mu}_{\lambda\lambda'} \equiv \langle p+\frac{1}{2}q,\lambda | J^{\mu}(0) | p-\frac{1}{2}q,\lambda\rangle$

$$= \bar{u}(p+\tfrac{1}{2}q,\lambda')\left[F_1\gamma^{\mu} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2\right]u(p-\tfrac{q}{2},\lambda)$$



a)

$$J^{\mu}(x) \equiv \bar{q}(x)\gamma^{\mu}q(x)$$

$$\gamma^0 = \tfrac{1}{2}(\gamma^+ + \gamma^-)$$

$$J^+(x) = q^\dagger\gamma^0\gamma^+ q$$

$$\gamma^+\gamma^+ = \not{n}\not{n} = 0$$

$$= \tfrac{1}{2}q^\dagger\gamma^-\gamma^+ q \equiv 2q^\dagger q_+ \qquad q_+ \equiv \tfrac{1}{4}\gamma^-\gamma^+ q$$

$$q_+^\dagger = \tfrac{1}{4}q^\dagger(\gamma^+)^\dagger(\gamma^-)^\dagger = \tfrac{1}{4}q^\dagger\gamma^-\gamma^+$$

$$\gamma^-\gamma^+\gamma^-\gamma^+ = \tilde{\not{n}}\not{n}\tilde{\not{n}}\not{n} = 2n\cdot\tilde{n}\,\tilde{\not{n}}\not{n} = 4\gamma^-\gamma^+$$

$$q_+^\dagger q_+ = \tfrac{1}{16}q^\dagger\gamma^-\gamma^+\gamma^-\gamma^+q = \tfrac{1}{4}q^\dagger\gamma^-\gamma^+q = \tfrac{1}{2}J^+(x)$$

b) F_1 contrib. to $F^+_{++} \equiv F^+_{1/2\,1/2}$: $\qquad q^+ = 0$, $\vec{p}_\perp = 0$

$$F^+_{++}(1) = \frac{F_1}{p^+}\mathrm{Tr}\left[(\not{p}+\tfrac{1}{2}\not{q}+m)\underbrace{\gamma^+}_{\not{n}}(\not{p}-\tfrac{1}{2}\not{q}+\overset{\circ}{m})\tfrac{1}{4}\not{n}(1+\overset{\circ}{\gamma_5})\right]$$

$$= F_1\frac{2n\cdot(p-\frac{1}{2}q)}{4p^+}\mathrm{Tr}\left[(\not{p}+\tfrac{1}{2}\not{q})\not{n}\right] = 2p^+F_1 = F^+_{--}(1)$$

F_2-contrib:

$$F^+_{++}(2) = \frac{iF_2}{2mp^+}\mathrm{Tr}\left[(\not{p}+\tfrac{1}{2}\not{q}+m)\tfrac{i}{2}(\not{n}\not{q} - \overbrace{\not{q}\not{n}}^{+\not{n}\not{q}\ \text{since}\ n\cdot q=0})(\not{p}-\tfrac{1}{2}\not{q}+m)\tfrac{1}{4}\not{n}(1+\overset{\circ}{\gamma_5})\right]$$

$$= -\frac{F_2}{8mp^+}\mathrm{Tr}\left[(\not{p}+\tfrac{1}{2}\not{q}+\overset{\circ}{m})\not{n}\not{q}(\not{p}-\tfrac{1}{2}\not{q}+\overset{\circ}{m})\not{n}\right] = 0 = F^+_{--}(2)$$

since m-terms vanish due to $n^2=0$ and $\not{n}\not{q} = -\not{q}\not{n}$, leaving an uneven # γ-matrices.

16 cont. F_1 contr. to F^+_{+-} :

$$F^+_{+-}(1) = \frac{F_1}{p^+} \mathrm{Tr}\left[(\not p + \tfrac{1}{2}\not q + m)\not n(\not p - \tfrac{1}{2}\not q + \overset{\circ}{m})\frac{-1}{2\sqrt{2}}\not n \not e_+\right]$$

$$= \frac{-mF_1}{2\sqrt{2}\,p^+}\mathrm{Tr}\left[\not n(\not p - \tfrac{1}{2}\not q)\not n \not e_+\right] = 0 \quad \text{since } n\cdot e_+ = 0$$

$$F^+_{+-}(2) = -\frac{i\cdot i\,F_2}{2\sqrt{2}\,2m\,p^+}\mathrm{Tr}\left[(\not p + \tfrac{1}{2}\not q + \overset{\circ}{m})\not n \not q(\not p - \tfrac{1}{2}\not q + \overset{\circ}{m})\not n \not e_+\right]$$

$$= \frac{F_2\cdot 2p^+}{4\sqrt{2}\,m\,p^+}\mathrm{Tr}\left[\not q(\not p - \tfrac{1}{2}\not q)\not n \not e_+\right] \quad \left|\ \begin{array}{l} n\cdot e_+ = 0 \\ p\cdot e_+ = 0 \\ n\cdot q = 0 \end{array}\right.$$

$$= \sqrt{2}\,\frac{F_2}{m}\,p^+\, e_+\cdot q = \frac{q_\perp}{m}e^{i\varphi}p^+F_2 \quad \left|\ \vec{q}_\perp = q_\perp(\cos\varphi, \sin\varphi)\right.$$

$$\Rightarrow \quad F^+_{\frac{1}{2}\frac{1}{2}} = F^+_{-\frac{1}{2}-\frac{1}{2}} = 2p^+F_1(Q^2)$$

$$F^+_{\frac{1}{2},-\frac{1}{2}} = \frac{q_\perp}{m}e^{i\varphi}p^+F_2(Q^2); \quad F^+_{-\frac{1}{2}\frac{1}{2}} = -\frac{q_\perp}{m}e^{-i\varphi}p^+F_2(Q^2)$$

c) $|s_\perp = \tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left[|\lambda = +\rangle + |\lambda = -\rangle\right]$

$$\langle p + \tfrac{1}{2}q, s_\perp = \tfrac{1}{2}|J^+|p - \tfrac{1}{2}q, s_\perp = \tfrac{1}{2}\rangle = \tfrac{1}{2}\left(F^+_{++} + F^+_{--} + F^+_{+-} + F^+_{-+}\right)$$

$$= 2p^+F_1(Q^2) + \frac{i\,q_\perp}{m}\sin\varphi\; p^+F_2(Q^2)$$

2. $\displaystyle \frac{P^+}{2\pi} \int_{-\infty}^{\infty} dy^- \, e^{i x p^+ y^-} \langle p + \tfrac{1}{2}\Delta, \lambda' | \bar{q}(-\tfrac{1}{2}y) \gamma^+ q(\tfrac{1}{2}y) | p - \tfrac{1}{2}\Delta, \lambda \rangle_{y^+ = y_\perp = 0}$

$$= \bar{u}(p + \tfrac{1}{2}\Delta, \lambda') \left[H(x, \xi, t) \gamma^+ + E(x, \xi, t) \frac{i \sigma^{+,\nu}}{2m} \Delta_\nu \right] u(p - \tfrac{1}{2}\Delta, \lambda)$$

$\int_{-\infty}^{\infty} dx$ on both sides: $\displaystyle \int_{-\infty}^{\infty} dx \, e^{i x p^+ y^-} = 2\pi \, \delta(p^+ y^-) = \frac{2\pi}{p^+} \delta(y^-)$

$$\Rightarrow \langle p + \tfrac{1}{2}\Delta, \lambda' | \bar{q}(0) \gamma^+ q(0) | p - \tfrac{1}{2}\Delta, \lambda \rangle = \bar{u}(p + \tfrac{1}{2}\Delta, \lambda') \, *$$

$$* \left[\int_{-1}^{1} dx \, H(x, \xi, t) \gamma^+ + \int_{-1}^{1} dx \, E(x, \xi, t) \frac{i \sigma^{+,\nu}}{2m} \Delta_\nu \right] u(p - \tfrac{1}{2}\Delta, \lambda)$$

On the lhs. we may boost (e.g.) to the $\Delta^+ = 0$ frame:
$\bar{q}(0) \gamma^+ q(0)$ transforms like the '+' component of a 4-vector, and $0 \to 0$: The argument is invariant.
Similarly on the rhs. $\bar{u} \gamma^+ u$ and $\bar{u} \sigma^{+,\nu} \Delta_\nu u$ transform like the '+'-components of 4-vectors.

$\Rightarrow$ The relation is valid in any frame, and since $\bar{q}(0) \gamma^+ q(0) = J^+(0)$ we must have

$$\int_{-1}^{1} dx \, H(x, \xi, t) = F_1(t)$$

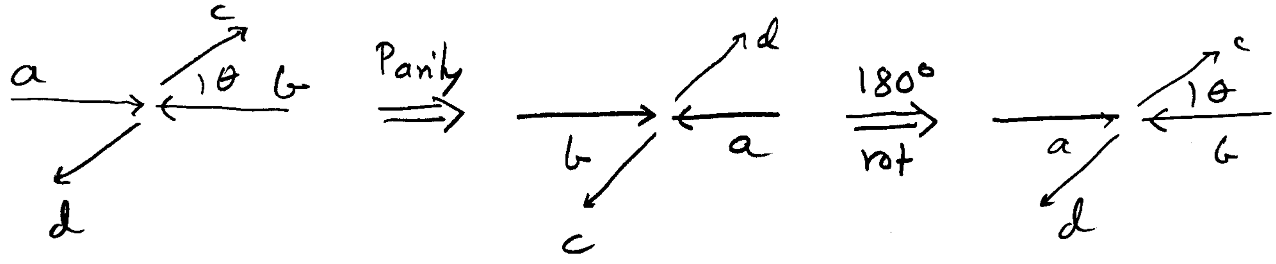
$$\int_{-1}^{1} dx \, E(x, \xi, t) = F_2(t)$$

3. a) Parity : $\vec{r} \to -\vec{r}$, $\vec{p} \to -\vec{p}$

$$\vec{L} = \vec{r} \times \vec{p} \to +\vec{L}$$

Helicity $\sim \vec{p}\cdot\vec{L} \to -\vec{p}\cdot\vec{L}$ reverses sign

Hence $T_{\lambda_a\lambda_b\lambda_c\lambda_d}(\theta,\varphi) = T_{-\lambda_a,-\lambda_b,-\lambda_c,-\lambda_d}(\theta,\varphi)$



Since the 4 momenta lie in a plane, parity can be undone with a 180° rotation along the normal to the plane.

But then $\sum\limits_{\lambda_{b,c,d}} |T_{\lambda_a\lambda_b\lambda_c\lambda_d}|^2 = \sum\limits_{\lambda_{b,c,d}} |T_{-\lambda_a\lambda_b\lambda_c\lambda_d}|^2$

and $\sigma(\lambda_a) = \sigma(-\lambda_a)$: $A_L = 0$

b) $|s_y = \pm\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left[|s_z = \frac{1}{2}\rangle \pm i\,|s_z = -\frac{1}{2}\rangle\right] \equiv \begin{cases} |\uparrow\rangle \\ |\downarrow\rangle \end{cases}$

$$A_y = \frac{|T_\uparrow|^2 - |T_\downarrow|^2}{|T_\uparrow|^2 + |T_\downarrow|^2} \qquad \left|\; \begin{aligned} T_\uparrow &= \frac{1}{\sqrt{2}}(T_+ + i T_-) \\ T_\downarrow &= \frac{1}{\sqrt{2}}(T_+ - i T_-) \end{aligned}\right.$$

$$|T_\uparrow|^2 = \frac{1}{2}\left(|T_+|^2 + |T_-|^2 - i T_+ T_-^* + i T_- T_+^*\right)$$

$$|T_\downarrow|^2 = \frac{1}{2}\left(|T_+|^2 + |T_-|^2 + i T_+ T_-^* - i T_- T_+^*\right)$$

36 cont.)

$$A_y = \frac{-i T_+ T_-^* + i T_- T_+^*}{|T_+|^2 + |T_-|^2} = \frac{2\,\mathrm{Im}(T_+ T_-^*)}{|T_+|^2 + |T_-|^2}$$

(sums over $\lambda_{b,c,d}$ understood for all TT^*'s)

Remark: $\sigma(s_a)$ is often expressed as an s_a-indep. term + a term linear in s_a. Consider a frame where $s_a^0 = 0$ (e.g., CM: $\vec{p}_a + \vec{p}_b = 0$).

Then $\sigma(s_a) = f_0(s,t) + \vec{s}_a \cdot \vec{p}_a \times \vec{p}_c\, f_1(s,t)$ is ok, since under parity:

$$\vec{s}_a \to +\vec{s}_a$$
$$\vec{p}_i \to -\vec{p}_i$$

No other term, e.g. $\vec{s}_a \cdot \vec{p}_i$, preserves P.

On the other hand $\vec{s}_a \cdot (\vec{p}_a \times \vec{p}_c)$ changes sign under time reversal:

$$\begin{cases} s_a \to -\vec{s}_a \\ \vec{p}_i \to -\vec{p}_i \end{cases}$$

This is why A_y requires amplitudes with a non-vanishing phase difference to preserve T-invariance.

with $\vec{s}_a = (0, s_a^y, 0)$ and $\vec{p}_a = (0, 0, p_a^z)$

$\vec{s}_a \cdot \vec{p}_a \times \vec{p}_c \propto p_c^x$: The SSA is seen as a difference between $\sigma(p_c^x > 0)$ and $\sigma(p_c^x < 0)$, at fixed $\vec{s}_a$.