QCD and Hadron Structure **Exam 16.1 2009**

Fall 2008 PTO for some formulas

1. Free quarks and gluons have not been observed in experiments. Discuss how the QCD lagrangian can nevertheless be tested experimentally:

   (a) Briefly explain what is meant by QCD factorization, and give an example.

   (b) Give an example of how the spin and number of quark colors can be tested experimentally.

   (c) Draw all Feynman diagrams up to $\mathcal{O}(g^1)$ of the hard subprocess amplitudes contributing to DIS, $ep \to eX$. Here g is the strong coupling ($\alpha_s = g^2/4\pi$).

2. (a) Briefly explain what is meant by an 'infrared safe observable".

   (b) The definition of thrust for an n-particle final state with momenta $p_1, \ldots, p_n$ is

$$\mathcal{T}_n = \max_{\boldsymbol{u}} \frac{\sum_{i=1}^{n} |\boldsymbol{p}_i \cdot \boldsymbol{u}|}{\sum_{i=1}^{n} |\boldsymbol{p}_i|} \quad \text{with} \quad |\boldsymbol{u}| = 1$$

   Show that $\mathcal{T}_n$ is infrared safe and determine its minimum and maximum value in the limit $n \to \infty$.

   (c) Why is thrust useful in comparing perturbative QCD calculations with data?

3. The quark transverse charge density in a hadron at transverse coordinate $\boldsymbol{b}_\perp$ is defined as

$$\rho(\boldsymbol{b}_\perp) = \int \frac{d^2\boldsymbol{q}_\perp}{(2\pi)^2} e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{b}_\perp} \frac{1}{2p^+} \langle p^+, \tfrac{1}{2}\boldsymbol{q}_\perp, \lambda| : J^+(0) : |p^+, -\tfrac{1}{2}\boldsymbol{q}_\perp, \lambda\rangle \tag{1}$$

   where $p^+ \equiv p^0 + p^3$ is the light-front momentum, $\boldsymbol{q}_\perp$ the transverse momentum and λ the helicity of the hadron, while $J^+(x) = \bar{\mathrm{q}}(x)\gamma^+\mathrm{q}(x)$ is the electromagnetic current operator.

   (a) Show that the current operator is a density, *i.e.*, that $J^+ = 2\mathrm{q}_+^\dagger \mathrm{q}_+$, and express q_+ in terms of the quark operator q.

   (b) Use (1) to evaluate the transverse charge density of an electron at lowest in α.

4. (a) Express the u, d-quark lagrangian $\mathcal{L}_\mathrm{q} = \sum_{f=u,d} \bar{\mathrm{q}}_f(i\not{D} - m_f)\mathrm{q}_f$ in terms of the right- and left-handed quark fields $\mathrm{q}_{R,L} \equiv \frac{1}{2}(1 \pm \gamma_5)\mathrm{q}$.

   (b) Show that when $m_u = m_d = 0$ the lagrangian has a global $U(2) \times U(2)$ symmetry.

   (c) Demonstrate that the currents $j^\mu_{R,L} = \bar{\mathrm{q}}_{R,L}\gamma^\mu\mathrm{q}_{R,L}$ and $j^{\mu,a}_{R,L} = \bar{\mathrm{q}}_{R,L}\gamma^\mu\sigma_a\mathrm{q}_{R,L}$ are conserved (at the classical level), where the Pauli matrices σ_a operate on the flavour spinors $\mathrm{q} = \binom{\mathrm{q}_u}{\mathrm{q}_d}$.

   (d) Which symmetries are conserved at the quantum level, and which are manifest in the physical hadron spectrum?

Useful formulas

The free electron field is

$$e_\alpha(x) = \int \frac{dp^+ d^2\boldsymbol{p}_\perp}{(2\pi)^3 2p^+} \sum_\lambda \left[b_{p,\lambda} u_\alpha(p,\lambda) e^{-ip\cdot x} + d^\dagger_{p,\lambda} v_\alpha(p,\lambda) e^{ip\cdot x} \right] \tag{2}$$

with $\left\{ b_{p,\lambda}, b^\dagger_{p',\lambda'} \right\} = \left\{ d_{p,\lambda}, d^\dagger_{p',\lambda'} \right\} = (2\pi)^3 2p^+ \delta(p^+ - p'^+) \delta^2(\boldsymbol{p}_\perp - \boldsymbol{p'}_\perp)$.

The light-front spinors are

$$\begin{aligned} u(p,\lambda) &= \frac{1}{\sqrt{p^+}} (\not{p} + m) \tilde{\chi}(\lambda) \\ \bar{u}(p,\lambda) &= \frac{1}{\sqrt{p^+}} \chi^\dagger(\lambda)(\not{p} + m) &(3) \\ v(p,\lambda) &= \frac{1}{\sqrt{p^+}} (\not{p} - m) \tilde{\chi}(-\lambda) \\ \bar{v}(p,\lambda) &= \frac{1}{\sqrt{p^+}} \chi^\dagger(-\lambda)(\not{p} - m) &(4) \end{aligned}$$

where

$$\tilde{\chi}(\pm)\chi^\dagger(\pm) = \frac{1}{4}\not{n}(1 \pm \gamma_5) \tag{5}$$

and $n = (n^+, n^-, \boldsymbol{n}_\perp) = (0, 2, \boldsymbol{0})$.

The Dirac matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with all the γ^μ. The trace of an odd number of γ-matrices vanishes and

$$\begin{aligned} tr(1) &= 4 &(6) \\ \mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} &(7) \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) &(8) \end{aligned}$$

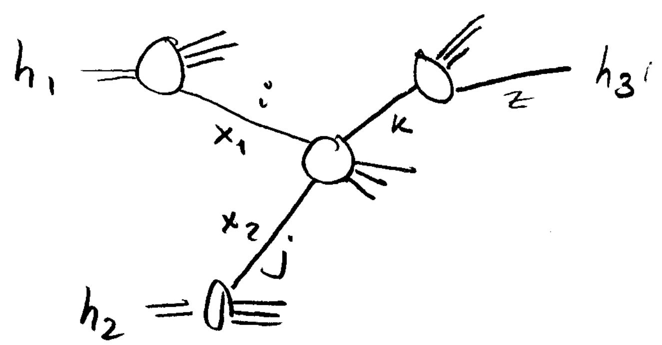
1\. a) QCD factorization implies that short- and long-distance phenomena are independent. A scattering cross-section such as $h_1 + h_2 \to h_3(p_\perp) + X$, where $p_\perp \gg \Lambda_{QCD}$ is a hard scale is then expressed as

$$E_3 \frac{d\vec{\sigma}}{d\vec{p}_3^{\,3}} = \int dx_1 dx_2\, f_1(x_1) f_2(x_2)\, \hat{\sigma}(ij \to k)\, F(z)$$

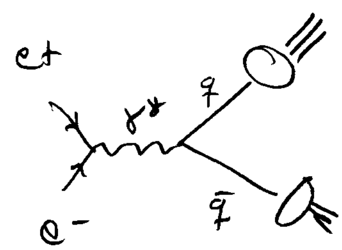
where f_1, f_2 are parton distributions of the initial hadrons h_1, h_2; $\hat{\sigma}$ is a hard subprocess cross section and $F(z)$ is a fragmentation function.

$f_1(x)$, $f_2(x)$, $F(z)$:
Universal, independent of hard subprocess

$\hat{\sigma}$: Calculable perturbatively



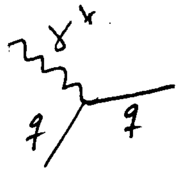
b) $e^+e^- \to q + \bar{q} \to 2$ jets



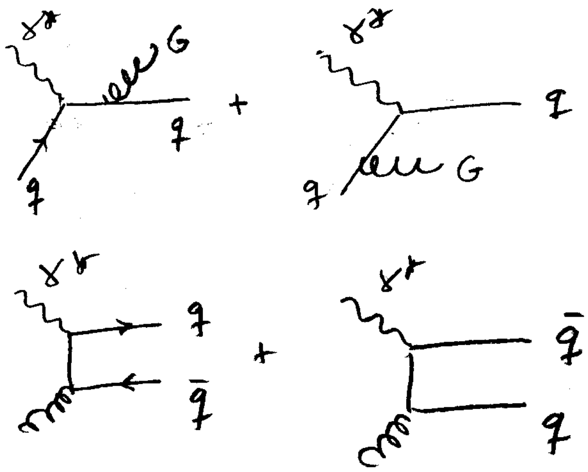
* The angular distribution of the jets $\propto 1 + \cos^2\theta$ reflects the spin ½ nature of the quarks

* $$\frac{\sigma(e^+e^- \to 2\,\text{jets})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_q e_q^2$$ $N_c = 3$ number of quark colors

1 c



$O(\alpha_s^0)$



2a An IRS observable is a cross section which is insensitive to

- Splitting any one hadron into 2 collinear hadrons

$$\xrightarrow[\vec{p}]{} \quad \Rightarrow \quad \xrightarrow[\vec{p}_1]{}\xrightarrow[\vec{p}_2]{} \qquad \vec{p} = \vec{p}_1 + \vec{p}_2 \ ; \ \vec{p}_1 \parallel \vec{p}_2$$

$$E_p \simeq |\vec{p}|; \quad E_i \simeq |\vec{p}_i|$$

- Addition of a soft hadron

$$\xrightarrow[\vec{p}]{} \quad \Rightarrow \quad \downarrow \vec{q} \xrightarrow{} \vec{p} \qquad |\vec{q}|, \ E_q \text{ small}$$

Under these changes the momenta of all other particles are kept essentially constant

Simplest example of IRS: σ_{tot}, since all final states are weighted equally

b)

$$T_n = \max_{\vec{u}} \frac{\sum_{i=1}^{n} |\vec{p}_i \cdot \vec{u}|}{\sum_{i=1}^{n} |\vec{p}_i|} \quad \text{is IRS, since}$$

$$\left.\begin{array}{l} \vec{p}\cdot\vec{u} = (\vec{p}_1 + \vec{p}_2)\cdot\vec{u} \\ |\vec{p}| = |\vec{p}_1| + |\vec{p}_2| \end{array}\right\} \text{ when } \vec{p}_1 = \lambda\vec{p}, \ \vec{p}_2 = (1-\lambda)\vec{p} \quad (0 < \lambda < 1)$$

and $|\vec{q}\cdot\vec{u}|$, $|\vec{q}|$ are small when $|\vec{q}|$ is small

$T_n \leq 1$ and $T_n = 1$ when all $\vec{p}_i \parallel \vec{u}$

The minimum value of T_n is reached when the $\vec{p}_i$ are isotropic in angle (spherical final state)

2b cont. Let all $|\vec{p}_i| = |\vec{p}|$, $|\vec{p}_i \cdot \vec{u}| = |\vec{p}| \cos\theta$

$$T_n^{min} = \frac{1}{4\pi} \int d\Omega \, |\cos\theta| = \frac{1}{2} \int_{-1}^{1} d\cos\theta \, |\cos\theta|$$

$$= \int_0^1 d\cos\theta \, \cos\theta = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} \leq T_n \leq 1$$

2c) Collinear and soft singularities of PQCD cancel in IRS observables. Hence the PQCD result in terms of quarks & gluons for, e.g., thrust, can be directly compared with the experimentally measured thrust distribution, determined from hadron momenta.

3 $$S(\vec{b}) = \int \frac{d^2\vec{q}_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{b}} \frac{1}{2p^+} \langle p^+, \tfrac{1}{2}\vec{q}_\perp, \lambda | J^+(0) | p^+, -\tfrac{1}{2}\vec{q}_\perp, \lambda\rangle$$

a) $$J^+ = \bar{q}\gamma^+ q = q^\dagger \gamma^0\gamma^+ q = \tfrac{1}{2} q^\dagger(\gamma^+ + \gamma^-)\gamma^+ q$$

$$= \tfrac{1}{2} q^\dagger \gamma^-\gamma^+ q \qquad \text{since } \gamma^+\gamma^+ = (\gamma^0+\gamma^3)(\gamma^0+\gamma^3)$$

$$= (\gamma^{0\,2} + \gamma^{3\,2} + \{\gamma^0, \gamma^3\}) = 0$$

$$= \tfrac{1}{2}(\gamma^+ q)^\dagger(\gamma^+ q)$$

$$= 2 q_+^\dagger q_+ \quad \text{with} \quad q_+ = \tfrac{1}{2}\gamma^+ q$$

3 b) $$e(x) = \int \frac{dp^+ d^2\vec{p}_\perp}{(2\pi)^3 2p^+} \sum_\lambda \left[b_{p,\lambda} u(p,\lambda) e^{-ip\cdot x} + d^\dagger_{p,\lambda} v(p,\lambda) e^{ip\cdot x}\right]$$

$$\{ b_{p,\lambda}, b^\dagger_{p',\lambda'} \} = (2\pi)^3 2p^+ \delta(p^+ - p'^+)\delta^2(\vec{p}_\perp - \vec{p}'_\perp)$$

$$|p^+, \vec{p}_\perp, \lambda\rangle = b^\dagger_{p,\lambda}|0\rangle$$

$$\langle p^+, \tfrac{1}{2}\vec{q}_\perp, \lambda | :\bar{e}(0)\gamma^+ e(0): | p^+, -\tfrac{1}{2}\vec{q}_\perp, \lambda\rangle =$$

$$= \bar{u}(p^+, \tfrac{1}{2}\vec{q}_\perp, \lambda)\gamma^+ u(p^+, -\tfrac{1}{2}\vec{q}_\perp, \lambda) =$$

Let $p \equiv (p^+, 0^-, \vec{0}_\perp)$
$q_\perp \equiv (0^+, 0^-, \vec{q}_\perp)$

$$u(p^+, -\tfrac{1}{2}\vec{q}_\perp, \lambda) = \frac{1}{\sqrt{p^+}} (\not{p} - \tfrac{1}{2}\not{q}_\perp + m)\, \tilde{\chi}(\lambda)$$

$$\bar{u}(p^+, \tfrac{1}{2}\vec{q}_\perp, \lambda) = \frac{1}{\sqrt{p^+}} \chi^+(\lambda)(\not{p} + \tfrac{1}{2}\not{q} + m)$$

$$\not{p} = \tfrac{1}{2} p^+ \gamma^- = \tfrac{1}{2} p^+ \tilde{\not{n}} \qquad \tilde{\chi}(\pm)\chi^+(\pm) = \tfrac{1}{4}\not{n}(1 \pm \gamma_5)$$

$$\gamma^+ = \not{n}$$

$$\bar{u}(p^+, \tfrac{1}{2}\vec{q}_\perp, \pm)\, \gamma^+\, u(p^+, -\tfrac{1}{2}\vec{q}_\perp, \pm) =$$

$$= \frac{1}{p^+} \mathrm{Tr}\Big[\big(\tfrac{1}{2} p^+ \tilde{\not{n}} + \overbrace{\tfrac{1}{2}\not{q}_\perp + m}^{\to 0} \big) \not{n} \big(\tfrac{1}{2} p^+ \tilde{\not{n}} \overbrace{- \tfrac{1}{2}\not{q}_\perp + m}^{\to 0 \text{ since } \not{n}\not{n}=0} \big) \tfrac{1}{4} \not{n}(1 \pm \gamma_5) \Big]$$

$$= \frac{p^+}{16} \mathrm{Tr}\big[\tilde{\not{n}} \underbrace{\not{n}\, \tilde{\not{n}}\, \not{n}}_{4} \big] = \frac{p^+}{4} 4\, \tilde{n}\cdot n = 2p^+$$

$$\rho_e(\vec{b}_\perp) = \int \frac{d^2\vec{q}_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp \cdot \vec{b}_\perp} \frac{1}{2p^+} \cdot 2p^+ = \delta^2(\vec{b}_\perp)$$

4 $$\mathcal{L}_q = \sum_{f=u,d} \bar{q}_f (i\not{D} - m_f) q_f$$

$$D^\mu = \partial^\mu + i g A^\mu_a \tfrac{1}{2}\lambda^a$$

a) $$q = \tfrac{1}{2}(1+\gamma_5)q + \tfrac{1}{2}(1-\gamma_5)q = q_R + q_L$$

$$\bar{q}_{R,L} = q^+_{R,L}\gamma^0 = q^+ \tfrac{1}{2}(1\pm\gamma_5)\gamma^0 = q^+\gamma^0 \tfrac{1}{2}(1\mp\gamma_5) = \bar{q}\tfrac{1}{2}(1\mp\gamma_5)$$

$$\bar{q}_R q_R = \tfrac{1}{4}\bar{q}(1-\gamma_5)(1+\gamma_5)q = 0 = \bar{q}_L q_L$$

$$\bar{q}_R q_L = \tfrac{1}{4}\bar{q}(1-\gamma_5)(1-\gamma_5)q = \tfrac{1}{2}\bar{q}(1-\gamma_5)q$$

$$\bar{q}_L q_R = \tfrac{1}{2}\bar{q}(1+\gamma_5)q$$

$$\Rightarrow \bar{q}q = \bar{q}_R q_L + \bar{q}_L q_R$$

$$\bar{q}_R \gamma^\mu q_R = \tfrac{1}{4}\bar{q}(1-\gamma_5)\gamma^\mu(1+\gamma_5) = \tfrac{1}{2}\bar{q}\gamma^\mu(1+\gamma_5)q$$

$$\bar{q}_L \gamma^\mu q_L = \tfrac{1}{2}\bar{q}\gamma^\mu(1-\gamma_5)q$$

$$\bar{q}_R \gamma^\mu q_L = \bar{q}_L \gamma^\mu q_R = 0$$

$$\mathcal{L}_q = \sum_f (\bar{q}_{R,f} + \bar{q}_{L,f})(i D_\mu \gamma^\mu - m_f)(q_{R,f} + q_{L,f})$$

$$= \sum_f \left[\bar{q}_{R,f} i\not{D} q_{R,f} + \bar{q}_{L,f} i\not{D} q_{L,f} - m_f(\bar{q}_{R,f} q_{L,f} + \bar{q}_{L,f} q_{R,f}) \right]$$

b) Let $q = \begin{pmatrix} q_u \\ q_d \end{pmatrix}$. Then for $m_u = m_d = 0$,

$$\mathcal{L}_q = \bar{q}_R i \not{D} q_R + \bar{q}_L i \not{D} q_L$$

is invariant under

$$q_R \to U_R(2)\, q_R$$
$$q_L \to U_L(2)\, q_L$$

where $U_R(2)$ and $U_L(2)$ are unitory 2×2 matrices

c) Eqs of motion: $(i\not{\partial} - g\not{A})\, q_R = (i\not{\partial} - g\not{A})\, q_L = 0$

$$\bar{q}_R(-i\overleftarrow{\not{\partial}} - g\not{A}) = \bar{q}_L(-i\overleftarrow{\not{\partial}} - g\not{A}) = 0$$

Conserved currents: $j^\mu_{\substack{R\\L}} = \bar{q}_{\substack{R\\L}} \gamma^\mu q_{\substack{R\\L}}$

$$j^{\mu,a}_{\substack{R\\L}} = \bar{q}_{\substack{R\\L}} \gamma^\mu \sigma_a q_{\substack{R\\L}}$$

Check: (suppress R, L)

$$i\partial_\mu j^\mu = (i\partial_\mu \bar{q})\gamma^\mu q + \bar{q}\gamma^\mu i\partial_\mu q =$$

$$= \bar{q}(i\overleftarrow{\not{\partial}} + g\not{A})q + \bar{q}(i\overrightarrow{\not{\partial}} - g\not{A})q = 0$$

Similarly $\partial_\mu j^{\mu,a} = 0$

d) $U(2) = U(1) \times SU(2)$ for R and L and equivalently currents R+L and R−L are conserved.

$U_A(1) = U_{R-L}(1)$ is broken by an anomaly

$U_V(1) = U_{R+L}(1)$ is conserved, expresses quark (or baryon) number conversation

$SU_V(2) = SU(2)_{R+L}$ is manifest as isospin symmetry of spectrum

$SU_A(2) = SU(2)_{R-L}$ is spontaneously broken: No parity partners

Chiral symmetry spont. broken