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LF spinors and polarization vectors

The LF spinors and polarization vectors are defined in Appendix B of
S. J. Brodsky, H-C. Pauli and S. S. Pinsky Phys. Rep. **301**, 299 (1998).
The notation is $p = (p^+, p^-, \boldsymbol{p})$, with $p^\pm = p^0 \pm p^3$ and transverse vectors in bold. The auxiliary light-like vectors $n, \tilde{n}$ which satisfy $n \cdot p = p^+$ and $\tilde{n} \cdot p = p^-$ are

$$\begin{aligned} n &= (0, 2, \mathbf{0}) \\ \tilde{n} &= (2, 0, \mathbf{0}) \end{aligned} \tag{1}$$

Photon polarization vectors

For helicity $\lambda = \pm 1$ define

$$e_\lambda = (0, 0, \boldsymbol{e}_\lambda) = (0, 0, -\lambda, -i)/\sqrt{2} = -e_{-\lambda}{}^* \tag{2}$$

With $\boldsymbol{k} = k_\perp(\cos\phi, \sin\phi)$ we have then

$$e_\lambda \cdot k = -\boldsymbol{e}_\lambda \cdot \boldsymbol{k} = \lambda \frac{k_\perp}{\sqrt{2}} \exp(i\lambda\phi) \tag{3}$$

For $k^2 > 0$ let the $\lambda = 0$ vector be proportional to k,

$$e_0(k) = -i \frac{k}{\sqrt{k^2}} = -e_0(k)^* \tag{4}$$

The photon polarization vectors defined as

$$\varepsilon_\lambda(k) = e_\lambda - \frac{e_\lambda \cdot k}{k^+} n \qquad (\lambda = \pm 1,\ 0) \tag{5}$$

then satisfy $k \cdot \varepsilon_\lambda(k) = 0$ and $\varepsilon_\lambda(k) = -\varepsilon_{-\lambda}(k)^*$. They are orthonormal and complete,

$$\begin{aligned} \varepsilon_\lambda(k) \cdot \varepsilon_{\lambda'}(k)^* &= -\delta_{\lambda,\lambda'} \\ \sum_{\lambda=\pm 1,0} \varepsilon^\mu_\lambda(k)\, \varepsilon^\nu_\lambda(k)^* &= -g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2} \end{aligned} \tag{6}$$

The $\lambda = 0$ polarization vector can also be expressed as ($\epsilon^{0123} = 1$)

$$\varepsilon^\mu_0(k) = \frac{1}{\sqrt{k^2}} \epsilon^{\mu\nu\rho\sigma} \varepsilon_{1\,\nu} \varepsilon_{-1\,\rho} k_\sigma \tag{7}$$

Using the identity

$$\gamma^-(1-\gamma_5)(\gamma^1 - i\gamma^2) = 0$$

we note

$$\begin{aligned} &\not{n}(1+\gamma_5)\not{\varepsilon}_- = 0 \qquad && \not{\tilde{n}}(1-\gamma_5)\not{\varepsilon}_- = 0 \\ &\not{n}(1-\gamma_5)\not{\varepsilon}_+ = 0 \qquad && \not{\tilde{n}}(1+\gamma_5)\not{\varepsilon}_+ = 0 \end{aligned} \tag{8}$$

Also:

$$\not{\tilde{n}}\not{n}\gamma_5 = 2(\gamma_5 + i\gamma^x\gamma^y)\ , \qquad \mathrm{Tr}\,[\not{\tilde{n}}\not{n}\gamma_5\not{p}\not{q}] = -8i(p^x q^y - p^y q^x) \tag{9}$$

Light-Front spinors

The LF spinors for helicity $s = \pm\frac{1}{2}$ are

$$\begin{aligned} u(p,s) &= \frac{1}{\sqrt{p^+}}\,(p^+ + \gamma^0 m + \gamma^0 \boldsymbol{\gamma}\cdot\boldsymbol{p})\chi(s) \\ v(p,s) &= \frac{1}{\sqrt{p^+}}\,(p^+ - \gamma^0 m + \gamma^0 \boldsymbol{\gamma}\cdot\boldsymbol{p})\chi(-s) \end{aligned} \tag{10}$$

with

$$\chi(+) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \chi(-) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \tag{11}$$

Defining $\tilde{\chi}(s) \equiv \gamma^0\chi(s)$, *i.e.*,

$$\tilde{\chi}(+) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}; \quad \tilde{\chi}(-) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{12}$$

and using

$$\gamma^+\tilde{\chi}(s) = 0 \tag{13}$$

we have

$$\begin{aligned} u(p,s) &= \frac{1}{\sqrt{p^+}}\,(\not{p} + m)\tilde{\chi}(s) \\ \bar{u}(p,s) &= \frac{1}{\sqrt{p^+}}\,\chi^\dagger(s)(\not{p} + m) \end{aligned} \tag{14}$$

$$\begin{aligned} v(p,s) &= \frac{1}{\sqrt{p^+}}\,(\not{p} - m)\tilde{\chi}(-s) \\ \bar{v}(p,s) &= \frac{1}{\sqrt{p^+}}\,\chi^\dagger(-s)(\not{p} - m) \end{aligned} \tag{15}$$

The u and v spinors are related through

$$\begin{aligned} \gamma_5\, v(p,s) &= (-1)^{s+\frac{1}{2}}\, u(p,-s) \\ \bar{v}(p,s)\,\gamma_5 &= (-1)^{s-\frac{1}{2}}\, \bar{u}(p,-s) \end{aligned} \tag{16}$$

Using the anticommutation relation

$$\left\{\gamma^+, \gamma^\mu\right\} = 2g^{+,\mu} \quad (= 4\delta(-,\mu)) \tag{17}$$

and the identity (13) we have (for any momentum p),

$$\begin{aligned} \gamma^+ u(p,s) &= 2\sqrt{p^+}\,\tilde{\chi}(s) \\ \gamma^+ v(p,s) &= 2\sqrt{p^+}\,\tilde{\chi}(-s) \end{aligned} \tag{18}$$

In evaluating matrix elements it is useful to define

$$U_{ss'} \equiv \tilde{\chi}(s)\chi^{\dagger}(s') \tag{19}$$

for which

$$\begin{aligned}
U_{++} &= \frac{1}{4}\not{n}(1+\gamma_5) \\
U_{--} &= \frac{1}{4}\not{n}(1-\gamma_5) \\
\\
U_{+-} &= -\frac{1}{2\sqrt{2}}\not{n}\not{\epsilon}_+ \\
U_{-+} &= -\frac{1}{2\sqrt{2}}\not{n}\not{\epsilon}_-
\end{aligned} \tag{20}$$

LF spinors quantized along $-z$ axis

The "inverted" spinors are defined as

$$u_-(p,s) = \frac{1}{\sqrt{p^-}}(\not{p}+m)\chi(+s) = \frac{1}{\sqrt{p^-}}(p^- + \gamma^0 m + \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p})\tilde{\chi}(+s) \tag{21}$$

$$v_-(p,s) = \frac{1}{\sqrt{p^-}}(\not{p}-m)\chi(-s) = \frac{1}{\sqrt{p^-}}(p^- - \gamma^0 m + \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p})\tilde{\chi}(-s) \tag{22}$$

so that

$$\bar{u}_-(p,s) = \frac{1}{\sqrt{p^-}}\tilde{\chi}^{\dagger}(+s)(\not{p}+m) = \frac{1}{\sqrt{p^-}}\chi^{\dagger}(+s)(p^- + \gamma^0 m - \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p}) \tag{23}$$

$$\bar{v}_-(p,s) = \frac{1}{\sqrt{p^-}}\tilde{\chi}^{\dagger}(-s)(\not{p}-m) = \frac{1}{\sqrt{p^-}}\chi^{\dagger}(-s)(p^- - \gamma^0 m - \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p}) \tag{24}$$

In calculating scattering amplitudes we need

$$\tilde{U}_{ss'} \equiv \chi(s)\tilde{\chi}^{\dagger}(s') = \gamma^0 U_{ss'}\gamma^0 \tag{25}$$

Using $\gamma^0\not{n}\gamma^0 = \not{\tilde{n}}$ we get

$$\begin{aligned}
\tilde{U}_{++} &= \frac{1}{4}\not{\tilde{n}}(1-\gamma_5) \qquad & \tilde{U}_{--} = \frac{1}{4}\not{\tilde{n}}(1+\gamma_5) \\
\tilde{U}_{+-} &= \frac{1}{2\sqrt{2}}\not{\tilde{n}}\not{\epsilon}_+ \qquad & \tilde{U}_{-+} = \frac{1}{2\sqrt{2}}\not{\tilde{n}}\not{\epsilon}_-
\end{aligned} \tag{26}$$

Defining also $V_{ss'} \equiv \chi(s)\chi^{\dagger}(s')$ we get

$$\begin{aligned}
V_{++} &= \frac{1}{8}\not{\tilde{n}}\not{n}(I+\gamma_5) \qquad & V_{--} = \frac{1}{8}\not{\tilde{n}}\not{n}(I-\gamma_5) \\
V_{+-} &= -\frac{1}{4\sqrt{2}}\not{\tilde{n}}\not{n}\not{\epsilon}_+ \qquad & V_{-+} = -\frac{1}{4\sqrt{2}}\not{\tilde{n}}\not{n}\not{\epsilon}_-
\end{aligned} \tag{27}$$

and for $\tilde{V}_{ss'} \equiv \tilde{\chi}(s)\tilde{\chi}^\dagger(s') = U_{ss'}\gamma^0$ we have

$$\begin{aligned} \tilde{V}_{++} &= \frac{1}{8}\not{n}\tilde{\not{n}}(I - \gamma_5) \qquad & \tilde{V}_{--} = \frac{1}{8}\not{n}\tilde{\not{n}}(I + \gamma_5) \\ \tilde{V}_{+-} &= \frac{1}{4\sqrt{2}}\not{n}\tilde{\not{n}}\not{e}_+ \qquad & \tilde{V}_{+-} = \frac{1}{4\sqrt{2}}\not{n}\tilde{\not{n}}\not{e}_- \end{aligned} \tag{28}$$