APPENDIX

A-1 METRIC

Metric tensor:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{A-1}$$

Derivatives with respect to contravariant (x^μ) or covariant (x_μ) coordinates are sometimes abbreviated as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} \tag{A-2}$$

Summation over repeated Lorentz (Greek) or space (Latin) indices is understood unless explicitly stated:

$$V \cdot W = V_\mu W^\mu = V^\mu W_\mu = g_{\mu\nu} V^\mu W^\nu = g^{\mu\nu} V_\mu W_\nu = V^0 W^0 - \mathbf{V} \cdot \mathbf{W} = V^0 W^0 - V^i W^i \tag{A-3}$$

A boldface letter denotes a three-vector or the three-dimensional part of a contravariant four-vector:

$$\mathbf{V} = \{V^i, i = 1, 2, 3\} = \{V_x, V_y, V_z\} \tag{A-4}$$

The only exception concerns the three-dimensional gradient

$$\boldsymbol{\nabla} = \{\nabla_x, \nabla_y, \nabla_z\} = \left(\frac{\partial}{\partial x^i} = \partial_i\right) = \left(-\partial^i = -\frac{\partial}{\partial x_i}\right) \tag{A-5}$$

The d'alembertian operator is

$$\Box = \partial^\mu \partial_\mu = \partial_0^2 - \boldsymbol{\nabla}^2 \tag{A-6}$$

and the four-momentum operator reads

$$p^\mu = i\partial^\mu = \{i\partial^0, -i\boldsymbol{\nabla}\} \tag{A-7}$$

Totally antisymmetric Levi-Civita tensor:

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \{\mu, \nu, \rho, \sigma\} \text{ is an even permutation of } \{0, 1, 2, 3\} \\ -1 & \text{if it is an odd permutation} \\ 0 & \text{otherwise} \end{cases} \tag{A-8}$$

$$\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma} \tag{A-9}$$

Useful identities:

$$\begin{aligned} \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\mu'\nu'\rho'\sigma'} &= -\det(g^{\alpha\alpha'}) \qquad \alpha = \mu, \nu, \rho, \sigma \\ & \qquad\qquad\qquad\quad \alpha' = \mu', \nu', \rho', \sigma' \\ \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu}{}^{\nu'\rho'\sigma'} &= -\det(g^{\alpha\alpha'}) \qquad \alpha = \nu, \rho, \sigma \\ & \qquad\qquad\qquad\quad \alpha' = \nu', \rho', \sigma' \\ \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu}{}^{\rho'\sigma'} &= -2(g^{\rho\rho'}g^{\sigma\sigma'} - g^{\rho\sigma'}g^{\rho'\sigma}) \\ \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho}{}^{\sigma'} &= -6g^{\sigma\sigma'} \\ \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} &= -24 \end{aligned} \tag{A-10}$$

Three-dimensional antisymmetric tensor:

$$\varepsilon^{ijk} = \varepsilon_{ijk} = 1 \qquad \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3)$$

A-2 DIRAC MATRICES AND SPINORS

The γ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{A-11}$$

with γ^0 hermitian, γ^i antihermitian, and are related to the β and α matrices through

$$\gamma^0 = \beta \qquad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha} \tag{A-12}$$

$$\begin{aligned} \gamma_5 = \gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\ &= -i\gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^3\gamma^2\gamma^1\gamma^0 = \gamma_5^\dagger \end{aligned} \tag{A-13}$$

$$\begin{aligned} \gamma_5^2 &= I \\ \{\gamma_5, \gamma^\mu\} &= 0 \end{aligned} \tag{A-14}$$

Commutator of γ matrices:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \tag{A-15}$$

$$\begin{aligned} \gamma^\mu\gamma^\nu &= g^{\mu\nu} - i\sigma^{\mu\nu} \\ [\gamma_5, \sigma^{\mu\nu}] &= 0 \end{aligned} \tag{A-16}$$

$$\begin{aligned} \gamma_5\sigma^{\mu\nu} &= \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \\ \gamma_5\gamma^0\boldsymbol{\gamma} &= \boldsymbol{\Sigma} \qquad \text{where} \quad \Sigma^i \equiv \tfrac{1}{2}\varepsilon_{ijk}\sigma^{jk} \end{aligned} \tag{A-17}$$

Hermitian conjugates:

$$\begin{aligned}\gamma^0\gamma^\mu\gamma^0 &= \gamma^{\mu\dagger}\\ \gamma^0\gamma_5\gamma^0 &= -\gamma_5^\dagger = -\gamma_5\\ \gamma^0(\gamma_5\gamma^\mu)\gamma^0 &= (\gamma_5\gamma^\mu)^\dagger\\ \gamma^0\sigma^{\mu\nu}\gamma^0 &= (\sigma^{\mu\nu})^\dagger\end{aligned} \tag{A-18}$$

For any two spinors ψ_1 and ψ_2 and any 4×4 matrix Γ,

$$(\overline{\psi}_1\Gamma\psi_2)^* = \overline{\psi}_2(\gamma_0\Gamma^\dagger\gamma_0)\psi_1 \tag{A-19}$$

while the corresponding identity for two anticommutating spin $\frac{1}{2}$ fields involves an extra minus sign.

Charge conjugation matrix:

$$\begin{aligned}C\gamma_\mu C^{-1} &= -\gamma_\mu^T\\ C\gamma_5 C^{-1} &= \gamma_5^T\\ C\sigma_{\mu\nu}C^{-1} &= -\sigma_{\mu\nu}^T\\ C(\gamma_5\gamma_\mu)C^{-1} &= (\gamma_5\gamma_\mu)^T\end{aligned} \tag{A-20}$$

Pauli matrices:

$$\sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{A-21}$$

Dirac representation:

$$\begin{aligned}\gamma^0 = \beta = \sigma^3 \otimes I &= \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\\ \boldsymbol{\alpha} = \sigma^1 \otimes \boldsymbol{\sigma} &= \begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}\\ \boldsymbol{\gamma} = \beta\boldsymbol{\alpha} = i\sigma^2 \otimes \boldsymbol{\sigma} &= \begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix}\\ \gamma_5 = \gamma^5 = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix} &= \sigma^1 \otimes I\\ \gamma^5\gamma^0 = -i\sigma^2 \otimes I &= \begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\\ \gamma^5\boldsymbol{\gamma} = -\sigma^3 \otimes \boldsymbol{\sigma} &= \begin{pmatrix}-\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}\\ \gamma^5\gamma^0\boldsymbol{\gamma} = \boldsymbol{\Sigma} = I \otimes \boldsymbol{\sigma} &= \begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix}\\ \sigma^{0i} = i\sigma^1 \otimes \sigma^i = i\alpha^i &= i\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix}\\ \sigma^{ij} = \varepsilon_{ijk} I \otimes \sigma^k = \varepsilon_{ijk}\Sigma^k &= \varepsilon_{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\\ C = i\gamma^2\gamma^0 = -i\sigma^1 \otimes \sigma^2 &= \begin{pmatrix}0 & -i\sigma^2\\ -i\sigma^2 & 0\end{pmatrix}\end{aligned} \tag{A-22}$$

$$C^T = C^\dagger = -C \qquad CC^\dagger = C^\dagger C = I \qquad C^2 = -I \tag{A-23}$$

Majorana representation:

$$
\begin{aligned}
\gamma^0 &= \beta = \sigma^1 \otimes \sigma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \\
\alpha^1 &= -\sigma^1 \otimes \sigma^1 = \begin{pmatrix} 0 & -\sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \\
\alpha^2 &= \sigma^3 \otimes I = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
\alpha^3 &= -\sigma^1 \otimes \sigma^3 = \begin{pmatrix} 0 & -\sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \\
\gamma^1 &= iI \otimes \sigma^3 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \\
\gamma^2 &= -i\sigma^2 \otimes \sigma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \\
\gamma^3 &= -iI \otimes \sigma^1 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} \\
\gamma_5 &= \gamma^5 = \sigma^3 \otimes \sigma^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \\
C &= -i\sigma^1 \otimes \sigma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \qquad \text{also satisfies (A-23)}
\end{aligned}
\tag{A-24}
$$

Relation with the Dirac representation:

$$
\gamma^\mu_{\text{Majorana}} = U\gamma^\mu_{\text{Dirac}}U^\dagger \qquad \text{with} \quad U = U^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} I & \sigma^2 \\ \sigma^2 & -I \end{pmatrix}
$$

Chiral representation:

$$
\begin{aligned}
\gamma^0 &= \beta = -\sigma^1 \otimes I = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \\
\boldsymbol{\alpha} &= \sigma^3 \otimes \boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \\
\boldsymbol{\gamma} &= i\sigma^2 \otimes \boldsymbol{\sigma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \\
\gamma_5 &= \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
C &= -i\sigma^3 \otimes \sigma^2 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \qquad \text{satisfies (A-23)} \\
\sigma^{0i} &= i\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\
\sigma^{ij} &= \varepsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}
\end{aligned}
\tag{A-25}
$$

Relation with the Dirac representation:

$$\gamma^\mu_{\text{chiral}} = U\gamma^\mu_{\text{Dirac}}U^\dagger \qquad \text{with} \qquad U = \frac{1}{\sqrt{2}}(1-\gamma_5\gamma_0) = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} \tag{A-26}$$

Contraction identities:

$$\begin{aligned}
\not{a}\not{b} &= a\cdot b - i\sigma_{\mu\nu}a^\mu b^\nu \\
\gamma^\lambda\gamma_\lambda &= 4 \\
\gamma^\lambda\gamma^\mu\gamma_\lambda &= -2\gamma^\mu \\
\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\lambda &= 4g^{\mu\nu} \\
\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\lambda &= -2\gamma^\rho\gamma^\nu\gamma^\mu \\
\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\lambda &= 2(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\nu\gamma^\mu\gamma^\sigma) \\
\gamma^\lambda\sigma^{\mu\nu}\gamma_\lambda &= 0 \\
\gamma^\lambda\sigma^{\mu\nu}\gamma^\rho\gamma_\lambda &= 2\gamma^\rho\sigma^{\mu\nu}
\end{aligned} \tag{A-27}$$

Traces:

$$\begin{aligned}
\operatorname{tr} I &= 4 \\
\operatorname{tr} \gamma^\mu &= 0 \\
\operatorname{tr} \gamma^5 &= 0
\end{aligned} \tag{A-28}$$

The trace of an odd product of γ^μ matrices vanishes:

$$\begin{aligned}
\operatorname{tr}(\gamma^5\gamma^\mu) &= 0 \\
\operatorname{tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} \\
\operatorname{tr}(\sigma^{\mu\nu}) &= 0 \\
\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^5) &= 0 \\
\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\
\operatorname{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= -4i\varepsilon^{\mu\nu\rho\sigma} = 4i\varepsilon_{\mu\nu\rho\sigma}
\end{aligned} \tag{A-29}$$

$$\begin{aligned}
\operatorname{tr}(\not{a}_1\not{a}_2\cdots\not{a}_{2n}) &= \operatorname{tr}(\not{a}_{2n}\cdots\not{a}_2\not{a}_1) \\
\operatorname{tr}(\not{a}_1\cdots\not{a}_{2n}) &= a_1\cdot a_2 \operatorname{tr}(\not{a}_3\cdots\not{a}_{2n}) - a_1\cdot a_3\operatorname{tr}(\not{a}_2\not{a}_4\cdots\not{a}_{2n}) + \cdots + a_1\cdot a_{2n}\operatorname{tr}(\not{a}_2\cdots\not{a}_{2n-1}) \\
&= 4\Sigma\varepsilon(a_{i_1}\cdot a_{j_1})\cdots(a_{i_n}\cdot a_{j_n})
\end{aligned} \tag{A-30}$$

ε is the signature of the permutation $i_1 j_1 \cdots i_n j_n$, and the sum runs over the $(2n)!/2^n n!$ different pairings satisfying $1 = i_1 < i_2 < \cdots < i_n$, $i_k < j_k$.

Dirac spinors u and v solutions of the Dirac equation

$$\begin{aligned}
(\not{p} - m)u^{(\alpha)}(p) &= 0 \\
(\not{p} + m)v^{(\alpha)}(p) &= 0
\end{aligned} \tag{A-31}$$

are functions of the on-shell momentum p, with $p^0 = E_p \equiv \sqrt{m^2 + \mathbf{p}^2}$ and are labeled by a polarization index $\alpha = 1, 2$.

Conjugate spinors:

$$\bar{u} = u^\dagger\gamma^0 \qquad \bar{v} = v^\dagger\gamma^0 \tag{A-32}$$

$$\begin{aligned}
\bar{u}^{(\alpha)}(p)(\not{p} - m) &= 0 \\
\bar{v}^{(\alpha)}(p)(\not{p} + m) &= 0
\end{aligned} \tag{A-33}$$

Normalization:

$$\bar{u}^{(\alpha)}(p)u^{(\beta)}(p) = \delta^{\alpha\beta}$$
$$\bar{v}^{(\alpha)}(p)v^{(\beta)}(p) = -\delta^{\alpha\beta} \tag{A-34}$$
$$\bar{v}^{(\alpha)}(p)u^{(\beta)}(p) = \bar{u}^{(\alpha)}(p)v^{(\beta)}(p) = 0$$

Density:

$$\bar{u}^{(\alpha)}(p)\gamma^0 u^{(\beta)}(p) = u^{\dagger(\alpha)}(p)u^{(\beta)}(p) = \bar{u}^{(\alpha)}(\tilde{p})u^{(\beta)}(p) = \frac{E_p}{m}\,\delta^{\alpha\beta}$$

$$\bar{v}^{(\alpha)}(p)\gamma^0 v^{(\beta)}(p) = v^{\dagger(\alpha)}(p)v^{(\beta)}(p) = -\bar{v}^{(\alpha)}(\tilde{p})v^{(\beta)}(p) = \frac{E_p}{m}\,\delta^{\alpha\beta} \tag{A-35}$$

$$\tilde{p} = (p^0, -\mathbf{p})$$

Projection operators over the positive and negative energy states:

$$\Lambda_+(p) = \frac{\not{p} + m}{2m} = \sum_{\alpha=1,2} u^{(\alpha)}(p) \otimes \bar{u}^{(\alpha)}(p)$$
$$\Lambda_-(p) = \frac{m - \not{p}}{2m} = -\sum_{\alpha=1,2} v^{(\alpha)}(p) \otimes \bar{v}^{(\alpha)}(p) \tag{A-36}$$

Projectors over a definite polarization state along a space-like four-vector n orthogonal to p, $n \cdot p = 0$:

$$u(p, n) \otimes \bar{u}(p, n) = \frac{\not{p} + m}{2m}\,\frac{1 + \gamma_5 \not{n}}{2}$$
$$-v(p, n) \otimes \bar{v}(p, n) = \frac{m - \not{p}}{2m}\,\frac{1 + \gamma_5 \not{n}}{2} \tag{A-37}$$

For comments on helicity states, see Sec. 2-2-1.

Gordon identities:

$$\bar{u}^{(\alpha)}(p)\gamma^\mu u^{(\beta)}(q) = \frac{1}{2m}\,\bar{u}^{(\alpha)}(p)[(p + q)^\mu + i\sigma^{\mu\nu}(p - q)_\nu]u^{(\beta)}(q)$$
$$\bar{u}^{(\alpha)}(p)\gamma^\mu\gamma^5 u^{(\beta)}(q) = \frac{1}{2m}\,\bar{u}^{(\alpha)}(p)[(p - q)^\mu\gamma^5 + i\sigma^{\mu\nu}(p + q)_\nu\gamma^5]u^{(\beta)}(q) \tag{A-38}$$

In particular:

$$\bar{u}^{(\alpha)}(p)\not{q}u^{(\beta)}(p) = \delta^{\alpha\beta}\,\frac{p \cdot q}{m}$$
$$u^{(\alpha)\dagger}(p)\boldsymbol{\alpha}u^{(\beta)}(p) = \delta^{\alpha\beta}\,\frac{\mathbf{p}}{m} \tag{A-39}$$

A-3 NORMALIZATION OF STATES, *S* MATRIX, UNITARITY, AND CROSS SECTIONS

Normalization of one-boson states:

$$\langle p \,|\, p' \rangle = 2\omega_p(2\pi^3)\delta^3(\mathbf{p} - \mathbf{p}') \tag{A-40}$$

with $\omega_p \equiv \sqrt{\mathbf{p}^2 + m^2}$ and polarization indices omitted.

One-fermion states:

$$\langle p | p' \rangle = \frac{\omega_p}{m} (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \tag{A-41}$$

(For massless fermions such as neutrinos, it is safer to use a normalization of the form (A-40) in intermediate computations).

S matrix and invariant scattering amplitude:

$$\begin{aligned} S &= I + iT \\ \langle f | T | i \rangle &= (2\pi)^4 \delta^4(P_f - P_i) \mathcal{T}_{fi} \end{aligned} \tag{A-42}$$

Differential cross section for the scattering from an initial state $i = \{1, 2\}$ involving no massive fermion into a final state $f = \{3, 4, \ldots, n\}$:

$$d\sigma = \frac{1}{4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2}} \frac{|\mathcal{T}_{fi}|^2}{S} d\tilde{p}_3 \cdots d\tilde{p}_n (2\pi)^4 \delta^4(P_i - P_f) \tag{A-43}$$

The factor S is

$$S = \prod_i k_i! \tag{A-44}$$

if there are k_i identical particles of species i in the final state. The measure $d\tilde{p}$ generally denotes

$$d\tilde{p} = \frac{d^3p}{(2\pi)^3 2\omega_p} \tag{A-45a}$$

except for massive fermions for which

$$d\tilde{p} = \frac{d^3p}{(2\pi)^3} \frac{m}{\omega_p} \tag{A-45b}$$

Accordingly, if the incident particles 1 and/or 2 are massive fermions, the expression (A-43) has to be multiplied by $2m_1$ and/or $2m_2$.

The formula (A-44) may have to be supplemented by an average over the initial polarizations and a summation over the final ones.

The decay rate $d\Gamma = d(\tau^{-1})$ of a particle of mass M into particles $3, 4, \ldots, n$ is given in its rest frame by the right-hand side of Eq. (A-43) with the flux factor $1/4[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2}$ replaced by $1/2M$. The same modifications as above are to be brought when fermions are present.

Differential cross section for two-body scattering $1 + 2 \to 3 + 4$ of nonidentical particles:

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s \mathbf{q}^2} |\mathcal{T}(s, t)|^2 \tag{A-46a}$$

or

$$\frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{|\mathbf{q}'|}{|\mathbf{q}|} \frac{1}{64\pi^2 s} |\mathcal{T}(s, t)|^2 \tag{A-46b}$$

in terms of the Mandelstam variables: $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, and of the initial and final center of mass momenta

$$\begin{aligned} 4\mathbf{q}^2 &= \frac{\lambda(s, m_1^2, m_2^2)}{s} = \frac{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}{s} \\ 4\mathbf{q}'^2 &= \frac{\lambda(s, m_3^2, m_4^2)}{s} \end{aligned} \tag{A-47}$$

Optical theorem: total cross section $i \to \cdots$ in terms of the imaginary part of the forward elastic amplitude $\mathcal{T}_{ii}(s, t = 0)$:

$$\sigma_{\text{tot}}(i) = \frac{\text{Im } \mathcal{T}_{ii}(s, t = 0)}{\lambda^{1/2}(s, m_1^2, m_2^2)} \tag{A-48}$$

Decomposition into partial wave amplitudes for spinless particles:

$$\mathcal{T}_{fi}(s, t) = 16\pi \sum_l (2l + 1)P_l(\cos\theta)\mathcal{T}^l_{fi}(s) \qquad \text{(A-49)}$$

with

$$4|\mathbf{q}||\mathbf{q}'|\cos\theta = t - u + \frac{(m_1^2 - m_2^2)(m_3^2 - m_4^2)}{s}$$

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

Unitarity below the inelastic threshold results in

$$\mathcal{T}^l(s) = \frac{s^{1/2}}{2|\mathbf{q}|} e^{i\delta_l(s)} \sin\delta_l(s) \qquad \text{(A-50)}$$

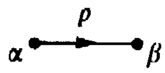
Generalization to particles with spin has been sketched in Chap. 5.

A-4 FEYNMAN RULES

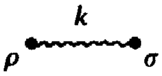
Feynman rules for the computation of a definite Green function or scattering amplitude:

1. Draw all possible topologically distinct diagrams—connected or disconnected but without vacuum-vacuum subdiagrams—contributing to the process under study, at the desired order.
2. For each diagram, and to each internal line, attach a propagator:

$$\frac{i}{k^2 - m^2 + i\varepsilon} \qquad \text{for a spin 0 boson} \qquad \text{(A-51)}$$



$$\left(\frac{i}{\not p - m + i\varepsilon}\right)_{\beta\alpha} \qquad \text{for a spin } \tfrac{1}{2} \text{ fermion} \qquad \text{(A-52)}$$



$$-i\left(\frac{g_{\rho\sigma} - k_\rho k_\sigma/\mu^2}{k^2 - \mu^2 + i\varepsilon} + \frac{k_\rho k_\sigma/\mu^2}{k^2 - \mu^2/\lambda + i\varepsilon}\right) \qquad \text{(A-53a)}$$

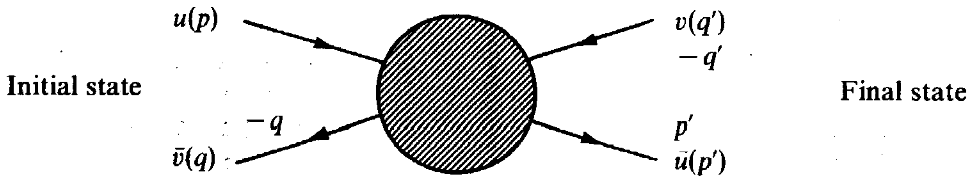
$$= -i\left[\frac{g_{\rho\sigma}}{k^2 - \mu^2 + i\varepsilon} - \frac{(1 - \lambda^{-1})k_\rho k_\sigma}{(k^2 - \mu^2 + i\varepsilon)(k^2 - \mu^2/\lambda + i\varepsilon)}\right] \qquad \text{(A-53b)}$$

for a spin 1 boson of mass μ in the Stueckelberg gauge, i.e., endowed with a kinetic lagrangian

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{\lambda}{2}(\partial\cdot A)^2 + \frac{\mu^2}{2}A^2$$

3. To each vertex, assign a weight derived from the relevant monomial of the interaction lagrangian. It is composed of a factor coming from the degeneracy of identical particles in the vertex, of the coupling constant appearing in $i\mathcal{L}_{\text{int}}$, of possible tensors in internal indices, and of a momentum conservation delta function $(2\pi)^4\delta^4(\Sigma p)$. To each field derivative $\partial_\mu\phi$ is associated $-ip_\mu$ where p is the corresponding incoming momentum. Vertices for the most common theories are listed below.
4. Carry out the integration over all internal momenta with the measure $d^4k/(2\pi)^4$, possibly after a regularization.
5. Multiply the contribution of each diagram by
   (a) a symmetry factor $1/S$ where S is the order of the permutation group of the internal lines and vertices leaving the diagram unchanged when the external lines are fixed;
   (b) a factor minus one for each fermion loop; and
   (c) a global sign for the external fermion lines, coming from their permutation as compared to the arguments of the Green function at hand (see Chap. 6).

These rules yield truncated functions with no factor on the external lines. Connected functions $(2\pi)^4\delta^4(\Sigma p)G_c(p_1,\ldots,p_n)$ are obtained by retaining only connected diagrams and by putting propagators (A-51) to (A-53) on the external lines. Contributions to proper Green functions $i(2\pi)^4\delta^4(\Sigma p)\Gamma(p_1,\ldots,p_n)$ come from one-particle irreducible diagrams. Finally, the scattering amplitude $i\mathscr{T}(2\pi)^4\delta^4(P_i - P_f)$ is obtained, up to renormalization, from the previous rules by putting the external lines on their mass shell, i.e., letting $p_i^2 = m_i^2$, and providing external fermion lines with spinors $u(p)$, $v(q')$, $\bar{u}(p')$, $\bar{v}(q)$ according to whether the line enters or leaves the diagram and whether it belongs to the initial or final state ($\not{p} = \not{p}' = m$, $\not{q} = \not{q}' = -m$).



Standard theories

(*a*) *φ^4 theory*

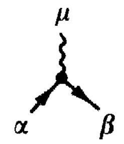
$$\mathscr{L} = \tfrac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda\varphi^4}{4!}$$

Propagator (A-51)
Vertex $\quad -i\lambda(2\pi)^4\delta^4(\Sigma p)$

(*b*) *Quantum electrodynamics*

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{\lambda}{2}(\partial\cdot A)^2 + \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi$$

Photon propagator (A-53*b*) with $\mu^2 = 0$
Fermion propagator (A-52)

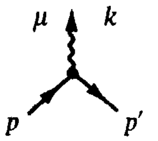
Vertex 

$-ie(\gamma_\mu)_{\beta\alpha}(2\pi)^4\delta^4(\Sigma p)$

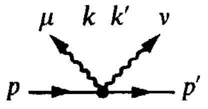
(*c*) *Scalar electrodynamics*

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{\lambda}{2}(\partial\cdot A)^2 + [(\partial_\mu + ieA_\mu)\varphi]^\dagger[(\partial^\mu + ieA^\mu)\varphi] - m^2\varphi^\dagger\varphi - \frac{g}{4}(\varphi^\dagger\varphi)^2$$

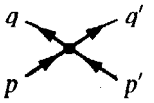
Photon propagator (A-53) with $\mu^2 = 0$
Scalar propagator (A-51) oriented along the charge flow
Vertices:



$-ie(p_\mu + p'_\mu)(2\pi)^4\delta^4(p - p' - k)$



$2ie^2 g_{\mu\nu}(2\pi)^4\delta^4(p - p' - k - k')$



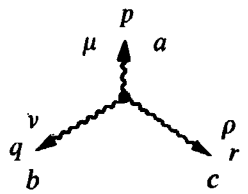
$-ig(2\pi)^4\delta^4(p + p' - q - q')$

(*d*) *Nonabelian gauge theory*

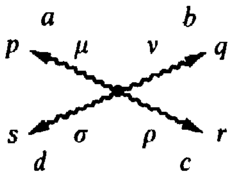
$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - gC_{abc}A_{\mu b}A_{\nu c})(\partial^\mu A^\nu{}_a - \partial^\nu A^\mu{}_a - gC_{abc}A^\mu_b A^\nu{}_c)$$
$$-\frac{\lambda}{2}(\partial_\mu A^\mu{}_a)^2 - \bar{\eta}_a\partial_\mu(\partial^\mu\delta_{ac} - gC_{abc}A^\mu{}_b)\eta_c$$
$$+\bar{\psi}[i\gamma_\mu(\partial^\mu - gA^\mu{}_a T^a) - m]\psi + [(\partial^\mu - gA^\mu{}_a T^a)\phi]^\dagger[(\partial_\mu - gA_{\mu a}T_a)\phi] - m_\phi^2\phi^\dagger\phi$$

Vector propagator as in (A-53*b*), with $\mu^2 = 0$
Ghost η propagator as in (A-51)
A minus sign for each ghost loop

Vertices:

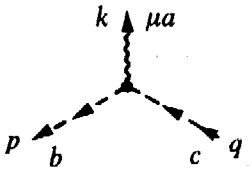


$$gC_{abc}(2\pi)^4\delta^4(p+q+r)[g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-r)_\mu + g_{\rho\mu}(r-p)_\nu]$$



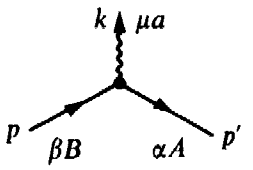
$$-ig^2(2\pi)^4\delta^4(p+q+r+s)[C_{eab}C_{ecd}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$
$$+ C_{eac}C_{edb}(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma}) + C_{ead}C_{ebc}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu})]$$

Ghost-vector vertex:



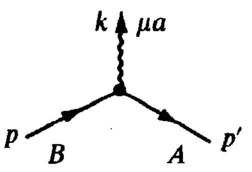
$$-gC_{abc}p_\mu(2\pi)^4\delta^4(k+p-q)$$

Fermion-vector vertex:

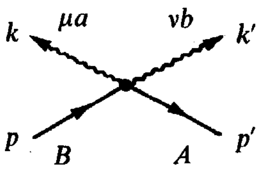


$$g(\gamma_\mu)_{\alpha\beta}T^a_{AB}(2\pi)^4\delta^4(p-p'-k)$$

Scalar-vector vertices:



$$gT^a_{AB}(p_\mu + p'_\mu)(2\pi)^4\delta^4(p-p'-k)$$



$$-ig^2 g_{\mu\nu}\{T^a, T^b\}_{AB}(2\pi)^4\delta^4(p-p'-k-k')$$

with T^a antihermitian.

Cabc = fabc

Ta -> -iTa