From QCD to Hadron Physics II

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Summary: first lecture

- low-energy ("strong") QCD not well-understood
  perturbative treatment impossible ($\alpha_s \gtrsim 1$)
- study this regime by investigating the symmetries
- limit of 3 massless quarks: chiral symmetry $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$
- phenomenology: chiral symmetry is spontaneously broken
  8 (nearly) massless pseudoscalar Goldstone bosons
- mass gap allows for the construction of effective field theory
  for these Goldstone bosons; effective Lagrangian:

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle \,, \quad U = \exp \frac{i\sqrt{2}}{F} \begin{pmatrix} \frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\phi_3}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\phi_8}{\sqrt{6}} \end{pmatrix}$$

$F \approx F_\pi$ pion decay constant

Lecture 2: Chiral perturbation theory

- A realistic effective theory for pions and kaons:
  - ▷ perturbation theory in the quark masses
  - ▷ quark mass ratios
  - ▷ isospin breaking and the inclusion electromagnetism
  - ▷ $\pi\pi$ scattering at leading order
- An effective quantum field theory:
  - ▷ loops, power counting, and all that
  - ▷ how we can still learn something from it all:

    quark mass ratios and $\eta \to 3\pi$

Explicit symmetry breaking: quark masses

- so far: chiral limit $m_u = m_d = m_s = 0$
  - ▷ theory for massless Goldstone bosons
  - ▷ $\mathcal{L}^{(2)} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle$ contains no terms $\propto M^2\phi^2$

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  - ▷ if symmetry breaking is weak, perform perturbative expansion in the quark masses "Chiral Perturbation Theory"

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  - ▷ if symmetry breaking is weak, perform
    perturbative expansion in the quark masses
    "Chiral Perturbation Theory"
- understand the transformation properties of the symmetry breaking term
- effective Lagrangian (properly generalised) still appropriate tool to systematically derive all symmetry relations

Quark masses in QCD

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}^0_{\text{QCD}} - \bar{q}\mathcal{M}q \, ,$$
$$\bar{q}\mathcal{M}q = \bar{q}_L \mathcal{M} q_R + h.c.$$

- quark mass term would be chirally invariant if $\mathcal{M}$ transformed according to

$$\boxed{\mathcal{M} \longmapsto \mathcal{M}' = L\mathcal{M}R^\dagger}$$

- assuming this, now construct invariant

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots, \mathcal{M})$$

- this procedure guarantees that chiral symmetry is broken in exactly the same way in the effective theory as it is in QCD

Leading order effective Lagrangian: masses (1)

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\left[\langle\partial_\mu U \partial^\mu U^\dagger\rangle + 2B\langle\mathcal{M}U^\dagger + \mathcal{M}^\dagger U\rangle\right]$$

- read off mass terms:

$$\begin{aligned} M^2_{\pi^\pm} &= B(m_u + m_d) \\ M^2_{K^\pm} &= B(m_u + m_s) \\ M^2_{K^0} &= B(m_d + m_s) \end{aligned}$$

- Gell-Mann–Oakes–Renner (GMOR) relation:

$$M^2_{GB} \propto m_q$$

- unified power counting for derivative / quark mass expansion:

$$m_q = \mathcal{O}(p^2)$$

Leading order effective Lagrangian: masses (2)

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\left[\langle \partial_\mu U \partial^\mu U^\dagger \rangle + 2B\langle \mathcal{M}U^\dagger + \mathcal{M}^\dagger U\rangle\right]$$

- flavour-neutral states ϕ_3, ϕ_8 are mixed:

$$\frac{B}{2}\begin{pmatrix}\phi_3\\ \phi_8\end{pmatrix}^T \begin{pmatrix} m_u + m_d & \frac{1}{\sqrt{3}}(m_u - m_d) \\ \frac{1}{\sqrt{3}}(m_u - m_d) & \frac{1}{3}(m_u + m_d + 4m_s)\end{pmatrix}\begin{pmatrix}\phi_3\\ \phi_8\end{pmatrix}$$

- diagonalise with

$$\begin{pmatrix}\pi^0\\ \eta\end{pmatrix} = \begin{pmatrix}\cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon\end{pmatrix}\begin{pmatrix}\phi_3\\ \phi_8\end{pmatrix}$$

- mixing angle $\quad \epsilon = \frac{1}{2}\arctan\left(\frac{\sqrt{3}}{2}\frac{m_d - m_u}{m_s - \hat{m}}\right)$, $\hat{m} = \frac{1}{2}(m_u + m_d)$

Leading order effective Lagrangian: masses (3)

- mass eigenvalues:

$$\begin{aligned} M^2_{\pi^0} &= B(m_u + m_d) - \mathcal{O}\big((m_u - m_d)^2\big) \\ M^2_\eta &= \frac{B}{3}(m_u + m_d + 4m_s) + \mathcal{O}\big((m_u - m_d)^2\big) \end{aligned}$$

- in the isospin limit $m_u = m_d$:

$$M^2_{\pi^\pm} = M^2_{\pi^0}\ , \quad M^2_{K^\pm} = M^2_{K^0} \quad \text{(of course!)}$$

- Gell-Mann–Okubo mass formula for pseudoscalars:

$$4M_K^2 = 3M_\eta^2 + M_\pi^2$$

$\Rightarrow$ fulfilled in nature at 7% accuracy

Quark mass ratios

- unknown factor B prevents quark mass determination
- quark mass ratios:

$$\frac{m_u}{m_d} = \frac{M_{K^+}^2 - M_{K^0}^2 + M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \approx 0.66$$

$$\frac{m_s}{m_d} = \frac{M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} \approx 22$$

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- m_u/m_d remarkable: why no large isospin violation in nature?
  - ▷ in purely pion physics, only $(m_d - m_u)^2$ occurs
  - ▷ $(m_d - m_u)/m_s$ (see $\pi^0\eta$ mixing angle) is small
  - ▷ $(m_d - m_u)/\Lambda_{\text{hadr}}$ is small

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- m_u/m_d remarkable: why no large isospin violation in nature?
- (strong) pion mass difference:

$$M^2_{\pi^0} = M^2_{\pi^+}\left\{1 - \frac{(m_d - m_u)^2}{8\hat{m}(m_s - \hat{m})} + \ldots\right\}$$

plug in quark mass ratios ...

$$M_{\pi^+} - M_{\pi^0} \approx 0.1\ \text{MeV}$$

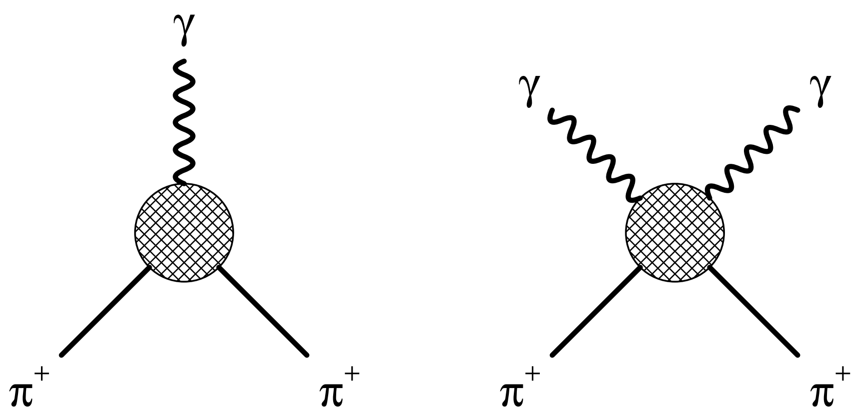
$$\text{vs.} \quad (M_{\pi^+} - M_{\pi^0})_{\text{exp}} \approx 4.6\ \text{MeV}$$

Electromagnetic effects

- coupling of $\mathcal{L}^{(2)}$ to external vector (v_μ) / axial vector (a_μ) currents via covariant derivative straightforward:

$$\partial_\mu U \longrightarrow D_\mu U = \partial_\mu U - i[v_\mu, U] - i\{a_\mu, U\}$$

- photons: $v_\mu = eA_\mu$, generates



(pion vector form factor, Compton scattering)

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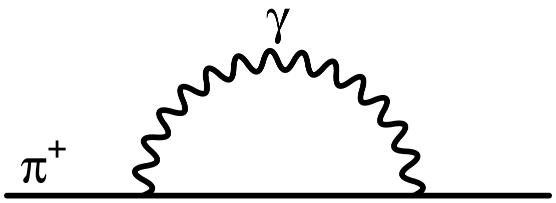

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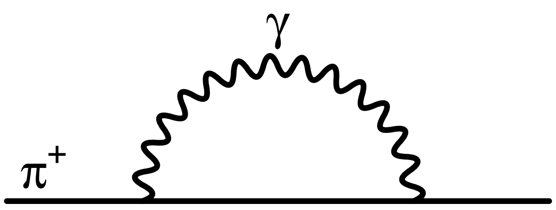
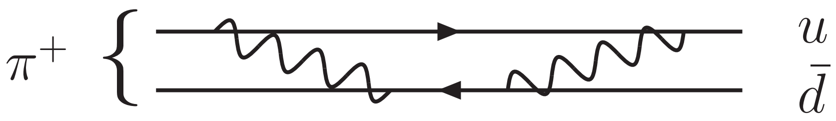
contains 

misses 

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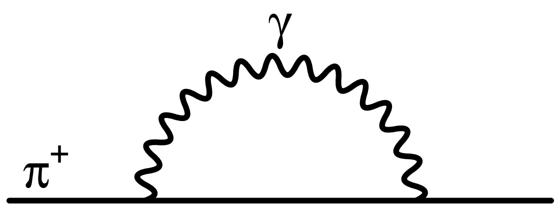

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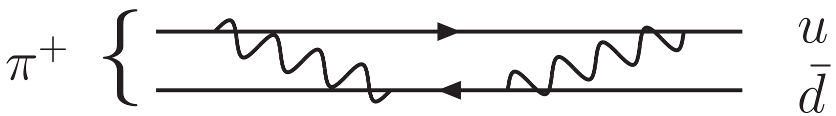

- include quark charges $Q = e\,\mathrm{diag}\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$
  power counting: $Q = \mathcal{O}(p)$
- one single term at $\mathcal{O}(e^2) = \mathcal{O}(p^2)$:

$$\boxed{\mathcal{L}^{(2)}_{\mathrm{em}} \;=\; C\langle QUQU^\dagger\rangle}$$

Electromagnetism in the meson masses

$$\mathcal{L}_{\text{em}}^{(2)} = C\langle QUQU^\dagger\rangle$$

- contribution to the meson masses:

$$M_{\pi^\pm}^2 = B(m_u+m_d) + \frac{2Ce^2}{F^2}$$
$$M_{K^\pm}^2 = B(m_u+m_s) + \frac{2Ce^2}{F^2}$$

  no contributions to neutral masses, $\pi^0\eta$-mixing

- Dashen's theorem:

$$\left(M_{\pi^+}^2 - M_{\pi^0}^2\right)_{\text{em}} = \left(M_{K^+}^2 - M_{K^0}^2\right)_{\text{em}}$$

  in the chiral limit!

- constant C fixed:

$$C = \frac{F_\pi^2}{2e^2}\left(M_{\pi^+}^2 - M_{\pi^0}^2\right)$$

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  in the chiral limit!

- improved quark mass ratio

$$\frac{m_u}{m_d} = \frac{M_{K^+}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 0.55 \quad \text{instead of } 0.66$$

$\pi\pi$ scattering to leading order

- $m_u = m_d$, $e^2 = 0$: isospin decomp. of $\pi\pi$ scattering amplitude:

$$M(\pi^a\pi^b \to \pi^c\pi^d) = \delta_{ab}\delta_{cd}A(s,t,u) + \delta_{ac}\delta_{bd}A(t,u,s) + \delta_{ad}\delta_{bc}A(u,s,t)$$

- calculate $A(s,t,u)$ from $\mathcal{L}^{(2)}$:

$$\boxed{A(s,t,u) \;=\; \frac{s - M_\pi^2}{F_\pi^2}}$$

Weinberg 1966

$\Rightarrow$ parameter-free prediction
$\Rightarrow$ in accordance with Goldstone theorem:
$\pi\pi$ interaction vanishes at low energies ($s \to 0$, $m_q \to 0$)

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- isospin amplitudes:

$$\begin{aligned} T^{I=0} &= 3A(s,t,u) + A(t,u,s) + A(u,s,t) \\ T^{I=1} &= A(t,u,s) - A(u,s,t) \\ T^{I=2} &= A(t,u,s) + A(u,s,t) \end{aligned}$$

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- s-wave scattering lengths: $a_0^I = \frac{1}{32\pi}T^I(s = 4M_\pi^2, t = u = 0)$

$$\boxed{a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16\ , \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045}$$

Chiral perturbation theory at higher orders

Why go beyond $\mathcal{O}(p^2)$?

- why not?? $\mathcal{O}(p^0)$ interactions forbidden by chiral symmetry, but all higher orders allowed
  $\Rightarrow$ consider $\mathcal{O}(p^4)$, $\mathcal{O}(p^6)$ ... corrections
- this is supposed to be a quantum field theory: what about loops?
- remember $\pi\pi$ scattering amplitude

$$A(s,t,u) = \frac{s - M_\pi^2}{F_\pi^2}$$

$\Rightarrow$ real! but unitarity requires (for partial waves t_ℓ^I)

$$\mathrm{Im} t_\ell^I = \sqrt{1 - \frac{4M_\pi^2}{s}} \left| t_\ell^I \right|^2$$

$\Rightarrow$ correct imaginary parts generated by loops

- how do loops feature in power counting?
- loops $\leftrightarrow$ divergences? renormalisation?

Weinberg's power counting argument (1)

- consider an arbitrary loop diagram based on

$$\mathcal{L}_{\text{eff}} = \sum_d \mathcal{L}^{(d)}$$

with L loops, I internal lines, V_d vertices of order d:

$$\mathcal{A} \propto \int (d^4p)^L \frac{1}{(p^2)^I} \prod_d (p^d)^{V_d}$$

- let $\mathcal{A}$ be of chiral dimension ν

$$\nu = 4L - 2I + \sum_d dV_d$$

- use topological identity for L to eliminate I

$$L = I - \sum_d V_d + 1 \quad \Rightarrow \quad \boxed{\nu = \sum_d V_d(d-2) + 2L + 2}$$

Weinberg's power counting argument (2)

$$\nu = \sum_d V_d(d-2) + 2L + 2$$

- chiral Lagrangian starts with $\mathcal{L}^{(2)}$, i.e. $d \geq 2$,
  i.e. right-hand-side is a sum of non-negative terms
  $\Rightarrow$ for fixed ν, there is only a finite number of combinations L, V_d
- $+2L$: each loop suppressed amplitude by two orders in the momentum expansion
- expansion in

$$\frac{M_\pi^2}{\Lambda_\chi^2} \quad \text{and} \quad \frac{p^2}{\Lambda_\chi^2}$$

where $\quad \Lambda_\chi \simeq M_{\text{res}} \simeq 4\pi F_\pi \simeq 1 \text{ GeV}$

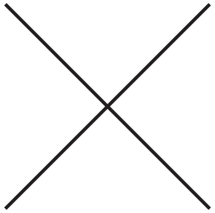
Weinberg's power counting argument (2)

$$\nu = \sum_d V_d(d-2) + 2L + 2$$

- example: $\pi\pi$ scattering

$\boxed{\nu = 2}$

only lowest-order tree graphs: $V_{d>2} = 0$, $L = 0$



Weinberg's power counting argument (2)

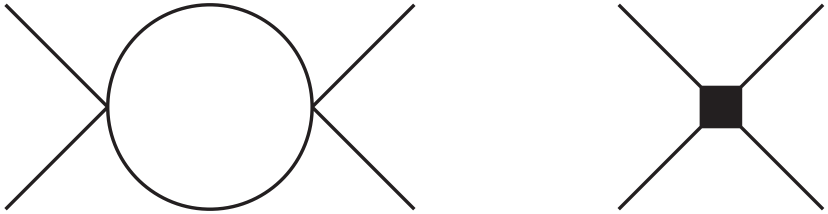
$$\nu = \sum_d V_d(d-2) + 2L + 2$$

- example: $\pi\pi$ scattering

$$\nu = 4$$

one-loop graphs with $\mathcal{L}^{(2)}$: $V_{d>2} = 0$, $L = 1$

or one insertion from $\mathcal{L}^{(4)}$: $V_4 = 1$, $V_{d>4} = 0$, $L = 0$



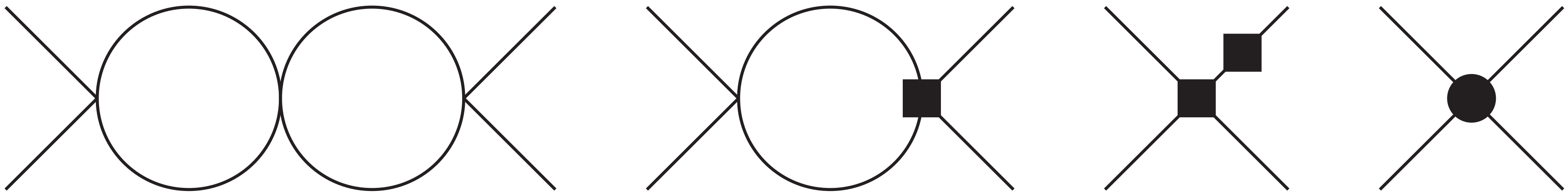
Weinberg's power counting argument (2)

$$\nu = \sum_d V_d(d-2) + 2L + 2$$

- example: $\pi\pi$ scattering

$\nu = 6$

two-loop graphs with $\mathcal{L}^{(2)}$: $V_{d>2} = 0,\ L = 2$
<u>or</u> one-loop with one vertex from $\mathcal{L}^{(4)}$: $V_4 = 1,\ V_{d>4} = 0,\ L = 1$
<u>or</u> two insertions from $\mathcal{L}^{(4)}$: $V_4 = 2,\ V_{d>4} = 0,\ L = 0$
<u>or</u> one insertion from $\mathcal{L}^{(6)}$: $V_4 = 0,\ V_6 = 1,\ V_{d>6} = 0,\ L = 0$



The Lagrangian $\mathcal{L}^{(4)}$ in SU(3)

$$
\begin{aligned}
\mathcal{L}^{(4)} &= L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
&+ L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+ L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + \chi U^\dagger)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
&+ L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
&- i\,L_9\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle \\
&+ \mathcal{L}_{\mathrm{WZW}}
\end{aligned}
$$

- $\chi = 2B(s+ip)$, $s = \mathcal{M} + \ldots$ (pseudo)scalar sources
  $F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]$, $F_L^{\mu\nu} = \ldots$ field strength tensors
  $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$ right-/left-handed currents
- $L_{1-3} = \mathcal{O}(\partial^4)$, $L_{4,5} = \mathcal{O}(\partial^2 m_q)$, $L_{6-8} = \mathcal{O}(m_q^2)$
- Wess–Zumino–Witten term / chiral anomaly $\mathcal{L}_{\mathrm{WZW}}$:
  - ▷ contains terms of odd intrinsic parity
  - ▷ describes processes such as

$$\pi^0 \to \gamma\gamma\ , \quad \eta \to \pi^+\pi^-\gamma^{(*)}\ , \quad K^+K^- \to \pi^+\pi^-\pi^0$$

Quark mass ratios revisited (1)

- calculate pion/kaon/eta masses beyond leading order:

$$M^2_{\pi^+} = B(m_u + m_d)\Big\{1 + \mathcal{O}(\hat{m}, m_s)\Big\}$$

$$M^2_{K^+} = B(m_u + m_s)\Big\{1 + \mathcal{O}(\hat{m}, m_s)\Big\}$$

- form dimensionless ratios:

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{m_u + m_d}\Big\{1 + \Delta_M + \mathcal{O}(m_q^2)\Big\}$$

$$\frac{(M^2_{K^0} - M^2_{K^+})_{\text{strong}}}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\Big\{1 + \Delta_M + \mathcal{O}(m_q^2)\Big\}$$

$$\Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8 - L_5) + \text{chiral logs}$$

- double ratio Q^2 particularly stable:

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{(M^2_{K^0} - M^2_{K^+})_{\text{strong}}}\Big\{1 + \mathcal{O}(m_q^2)\Big\}$$

Quark mass ratios revisited (2)

- Leutwyler's ellipse:

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$

Quark mass ratios revisited (2)

- Leutwyler's ellipse:

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$

- remember Dashen's theorem:

$$(M_{K^+}^2 - M_{K^0}^2)_{\text{em}} = (M_{\pi^+}^2 - M_{\pi^0}^2)_{\text{em}} + \mathcal{O}(e^2 m_q)$$

$$\boxed{Q_{\text{Dashen}} = 24.2}$$

Quark mass ratios revisited (2)

- Leutwyler's ellipse:

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2}\left(\frac{m_s}{m_d}\right)^2 = 1$$

- remember Dashen's theorem:

$$(M_{K^+}^2 - M_{K^0}^2)_{\text{em}} = (M_{\pi^+}^2 - M_{\pi^0}^2)_{\text{em}} + \mathcal{O}(e^2 m_q)$$

$$\boxed{Q_{\text{Dashen}} = 24.2}$$

- are $\mathcal{O}(e^2 m_q)$ corrections big?

$$1 \lesssim (M_{K^+}^2 - M_{K^0}^2)_{\text{em}}/(M_{\pi^+}^2 - M_{\pi^0}^2)_{\text{em}} \lesssim 2.5$$

$$\boxed{20.6 \lesssim Q \lesssim 24.2}$$

Where else to get information on Q^2? — $\eta \to 3\pi$

- η: isospin $I = 0$
  3 pions with angular momentum 0 cannot have $I = 0$, but $I = 1$
  $\Rightarrow$ $\eta \to 3\pi$ is an isospin violating process!
- use leading chiral Lagrangian (including isospin breaking!)

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \rangle + C\langle QUQU^\dagger \rangle$$

tree amplitude for $\eta \to \pi^+\pi^-\pi^0$

$$A(s,t,u) = \frac{B(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left\{1 + \frac{3(s-s_0)}{M_\eta^2 - M_\pi^2}\right\}$$

where $s = (p_\eta - p_{\pi^0})^2$, $t = (p_\eta - p_{\pi^+})^2$, $u = (p_\eta - p_{\pi^-})^2$,
$3s_0 = s + t + u = M_\eta^2 + 2M_{\pi^+}^2 + M_{\pi^0}^2$

- for $\eta \to 3\pi^0$, find $A(s,t,u) + A(t,u,s) + A(u,s,t)$

$\eta \to 3\pi$ (2)

$$A(s,t,u) = \frac{B(m_u - m_d)}{3\sqrt{3}F_\pi^2}\left\{1 + \frac{3(s-s_0)}{M_\eta^2 - M_\pi^2}\right\}$$

- electromagnetic term $C\langle QUQU^\dagger\rangle$ does not contribute

  Sutherland 1967

- contributions of order $e^2 m_q$ very small Baur, Kambor, Wyler 1995

  $\Rightarrow$ (potentially) much cleaner access to $m_u - m_d$

- rewrite $A(s,t,u)$ in terms of Q^2:

$$A(s,t,u) = \frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\left(M_\pi^2 - M_K^2\right)\frac{M(s,t,u)}{3\sqrt{3}F_\pi^2}$$

$$M(s,t,u) = \frac{3s - 4M_\pi^2}{M_\eta^2 - M_\pi^2} \qquad \text{(at leading order)}$$

$\eta \to 3\pi$ (3)

Problems:

- strong final-state interactions among pions (3 of them ...)

$$\text{tree:} \quad \Gamma(\eta \to \pi^+\pi^-\pi^0) = 66 \text{ eV}$$
$$\text{one-loop:} \quad \Gamma(\eta \to \pi^+\pi^-\pi^0) = 160 \pm 50 \text{ eV}$$

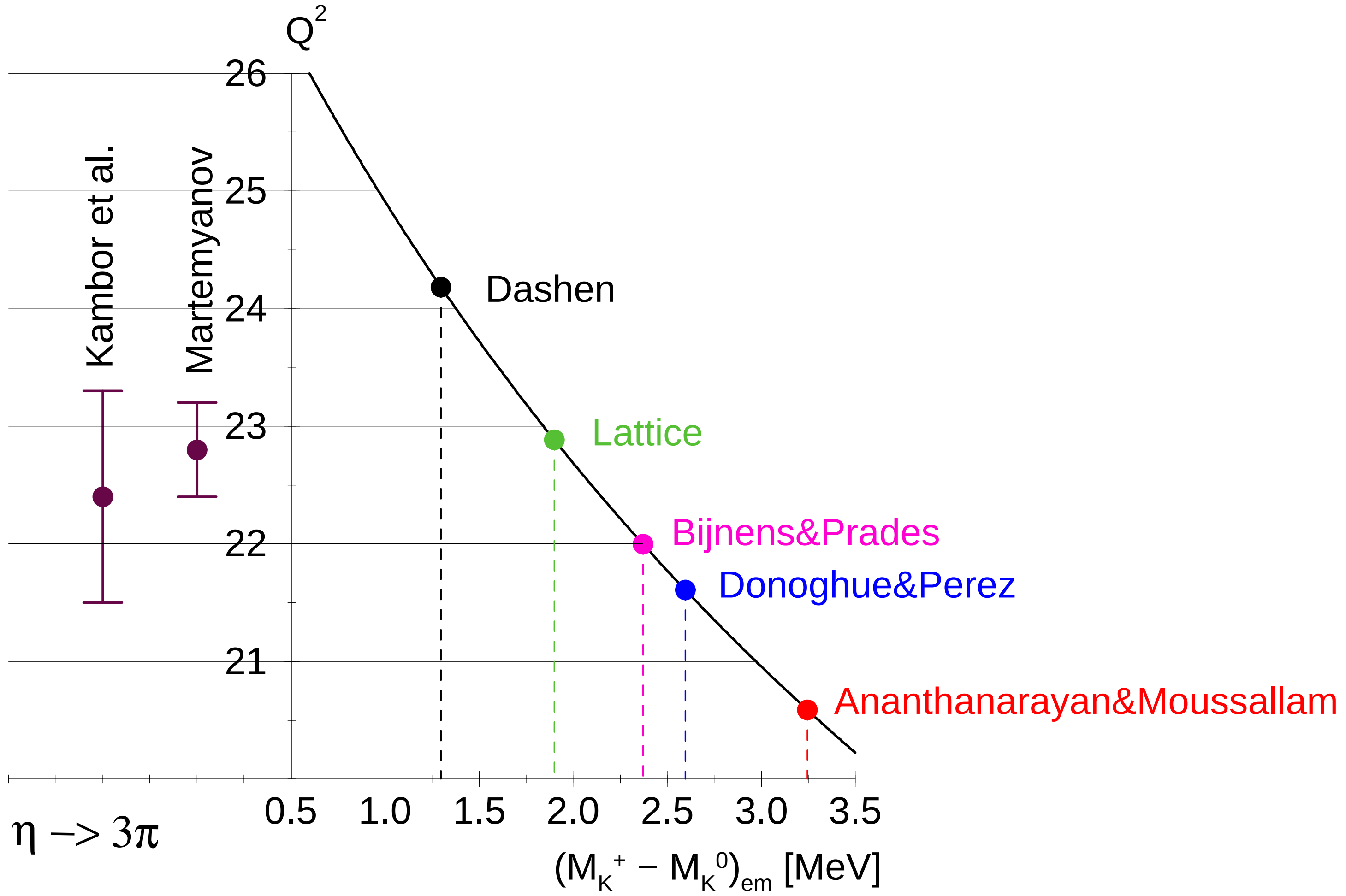
$\Rightarrow$ ongoing experimental efforts (for both final states)

KLOE, WASA@COSY, MAMI

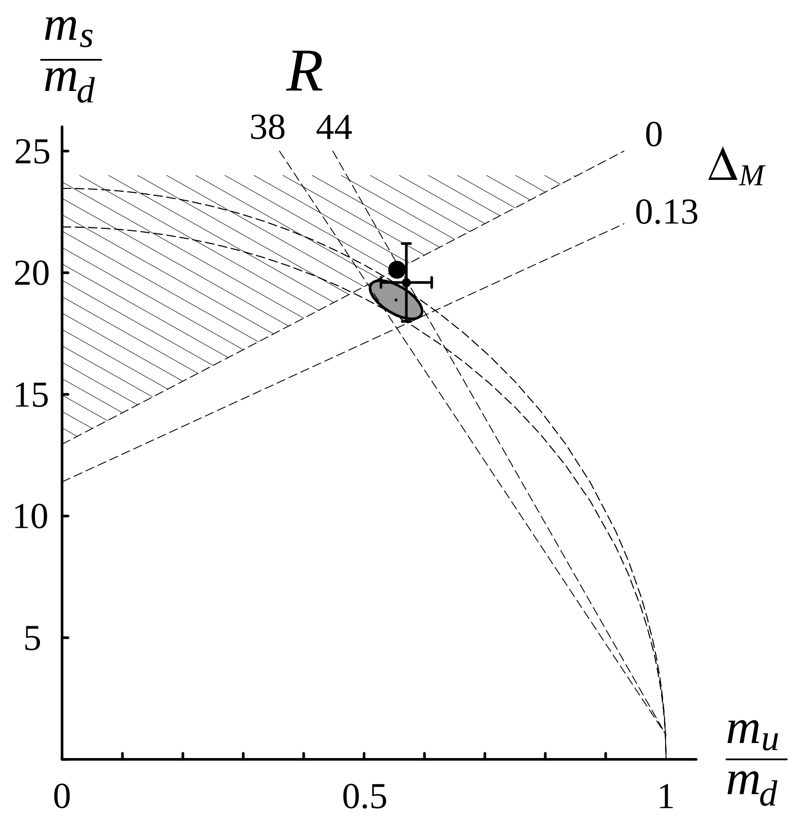
Combined result on quark mass ratios (1)

Combined information on Q

$\eta \to 3\pi$ vs. various corrections to Dashen's theorem:



Combined result on quark mass ratios (2)



Leutwyler 1996

- additional constraints needed to find position on the ellipse:
- $\eta\eta'$ mixing $\leftrightarrow$ large–N_c $\Rightarrow$ Δ_M
- $R = \dfrac{m_s - \hat{m}}{m_d - m_u}$ from baryon masses — Gasser, Leutwyler 1982

Summary: second lecture

- effective theory for Goldstone bosons as double expansion in momenta and quark masses
  $\longrightarrow$ chiral perturbation theory
- symmetry breaking pattern due to quark masses strictly equivalent to QCD
- can include electromagnetic effects (isospin breaking)
- ChPT is a quantum field theory with calculable higher-order (loop) corrections
- low-energy constants don't make it useless: they interrelate different processes / observables
  - ▷ various quark mass combinations $\leftrightarrow$ $\eta \to 3\pi$