Introduction and motivation

Quantum field theory (QFT) is the application of quantum mechanics (QM) to dynamical systems of <u>fields</u>, in the same sense that basic quantum mechanics deals mainly with the quantization of dynamical systems of <u>particles</u>.

The successful merging of the ideas of Special Relativity and Quantum mechanics could not be achieved through the relativistic extension of QM.

i) A practical reason:

When we write a single-particle relativistic wave equation (e.g. Klein-Gordon or Dirac equation), the theory gives rise to negative-energy states and negative probabilities (or negative norm states). This is an inconsistency which cannot be explained within relativistic quantum mechanics.

ii) A conceptual reason:

Relativistic quantum mechanics cannot describe systems with a variable number of particles. However, the quantum mechanical uncertainty principle, $\Delta E \cdot \Delta t \sim \hbar$, states that the energy can fluctuate wildly over a small interval of time. According to special relativity, energy can be converted into mass and vice versa. With quantum mechanics and special relativity, the wildly fluctuating energy can metamorphose into mass, i.e. into <u>new particles</u> not previously present. Such particles can exist only for a very short time, however arbitrary numbers of such "virtual" particles can be created. Moreover, Einstein's relation $E = mc^2$ allows for the actual creation of particle-antiparticle pairs.

<u>An illuminating example</u>

Consider a system of n particle in a nonstationary quantum state $|\psi\rangle$. Let $\{|\varphi_i\rangle\}$ be the complete set of eigenvectors of the operator $\hat{A}$, corresponding to the observable A, with discrete spectrum. Then $|\psi\rangle$ can be written as:

$$|\psi\rangle = \sum_i c_i |\varphi_i\rangle \quad , \quad \langle\varphi_i|\varphi_j\rangle = \delta_{ij}$$

The usual quantum mechanical interpretation is that $c_i^* c_i$ is the probability that, at a moment t, a number of n_i particles are in the state $|\varphi_i\rangle$. Since the state $|\psi\rangle$ is nonstationary, the probability changes during the process, so that after an interval δt, at least one of the particles which were initially in the state $|\varphi_i\rangle$ goes into another state, $|\varphi_k\rangle$.

Since

$$n = \langle\psi|\psi\rangle = \sum_{i,j} \langle\varphi_i| c_i^* c_j |\varphi_j\rangle =$$

$$= \sum_{i,j} c_i^* c_j \underbrace{\langle\varphi_i|\varphi_j\rangle}_{\delta_{ij}} = \sum_i |c_i|^2,$$

it follows that

$$n_i = |c_i|^2$$

and

$$\sum_i |c_i|^2 = \sum_i n_i = n .$$

Suppose that one particle goes from the state $|\varphi_i\rangle$ to $|\varphi_k\rangle$.

Then

$$n_k = c_k^* c_k \longrightarrow n_k + 1 .$$

$$n_i = c_i^* c_i \longrightarrow n_i - 1$$

In other words, c_i varies discretely and so does $|\psi\rangle$. However, in QM the state vector $|\psi\rangle$, as well as the coefficients c_i, are considered continuous and even differentiable functions.

It is therefore clear that within QM the systems with a variable number of particles cannot be consistently described.

Chronology of Quantum Mechanics and the beginning of Quantum Field Theory

1900 M. Planck (b. 1858) – explains black body radiation law
$E_\nu = h\nu$

1905 A. Einstein (b. 1879) – photoelectric effect

1913 N. Bohr (b. 1885) – hydrogen atom planetary model
→ justifies Rydberg's formula

1923-24 L. de Broglie (b. 1892) – concept of "associated wave" (to a parti
– "wave-particle" duality

1925 W. Heisenberg (b. 1901) – "MATRIX MECHANICS"
– quantization of the harmonic oscillator

Uhlenbeck & Goudsmith – spin of electron

W. Pauli (b. 1900) – Exclusion Principle

1926 E. Schrödinger (b. 1887) – "WAVE MECHANICS"
– proves equivalence of wave- and matrix mechanics

W. Gordon – introduces the relativistically invarian
wave function for the free scalar particle

M. Born (b.1882), W. Heisenberg, P. Jordan (b.1902) – quantization of the free radiation field

1927 O. Klein (b. 1894) – couples the electromagn. field to Gordon's equation

P.A.M. Dirac (b. 1902) – quantization of the electromagn. field (Spontaneous emission)

- Relativistic Quantum Mechanics in trouble !

- Brief detour to nonrelativistic QM

$$\hat{E}\psi = \hat{H}\psi$$ – Schrödinger eq.

$$H = \frac{p^2}{2m} + V$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \quad p^i = -i\hbar\frac{\partial}{\partial x^i}$$

$$\Downarrow$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi$$

$\rho = \psi^*\psi$ – probability density, POSITIVE-DEFINITE

– satisfies continuity eq.

$$\frac{\partial\rho}{\partial t} = -\nabla\vec{j}, \quad \vec{j} = \frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi)$$

$\Rightarrow$ conservation of probability $\Rightarrow$ conservation of NUMBER of PARTICLES in QM

sol. $\psi = Ce^{-iEt}$, $E > 0$!

- Relativistic QM $\to$ "Klein-Gordon eq.

– start from relativistic energy-momentum dispersion relation

$$E^2 = p^2 + m^2, \quad (\hbar^2 = c^2 = 1, \text{ natural units})$$

QM : $\hat{E}^2 = \hat{p}^2 + m^2$

$$\Downarrow$$

$$\boxed{(\Box + m^2)\psi = 0}, \quad \text{K-G. eq.}$$

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$ – d'Alembertian operator

$\Box$, m^2 – relativistically invariant $\Rightarrow$

$\Rightarrow$ ψ – relativistically invariant – describes spin 0 particle

$\Rightarrow$ $\psi^*\psi$ – relativistically invariant

- cont. eq. in relativistic case:

$$\partial_\mu j^\mu = 0 \quad , \quad j^\mu = (\rho, \vec{j})$$

Since j^μ is a 4-vector, its 0^{th} component ρ has to vary under Lorentz transformations. Since $\psi^*\psi$ is invariant, it follows that the probability density ρ CANNOT be $\psi^*\psi$.

$$\rho = \frac{i}{2m}\left[\psi^* \partial_t \psi - (\partial_t \psi)^* \psi\right] \text{ - NOT POSITIVE DEFINITE!}$$

$$\vec{j} = -\frac{i}{2m}\left[\psi^* \nabla \psi - (\nabla \psi)^* \psi\right]$$

- Sol. of K-G eq. $\sim e^{-ip_0 t}$, $p_0 = \pm E$, $E = \sqrt{\vec{p}^2 + m^2} > 0$

i.e. $\underset{\text{pos. en. sol}}{e^{-iEt}}$ AND $\boxed{\underset{\text{neg. en. sol}}{e^{+iEt}}}$!!

- negative en. sol. cannot be discarded as non-physical, due to the requirement of completeness of the set of solutions in QM. Moreover, for a particle which interacts electromagnetically, transitions between positive and negative-energy states are possible.

1928 P.A.M. Dirac (b. 1902) - relativistically covariant equation of a particle with spin 1/2

- Dirac linearizes the (operatorial) square-root

$$(\hat{p}^2 + m^2)^{1/2}$$

to obtain a differential eq. linear in time derivative (consequently also in space derivatives, due to relativistic covariance)

- the minimal no. of dimensions in which this linearization is possible is 4.

$(i\partial^\mu \gamma_\mu + m)\psi = 0$ - Dirac eq. for a free spin 1/2 particle

$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \mathbb{1}_{4\times4}$ - 4x4 matrices (constant!)

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix}$$ - SPINOR

- in the free case, ψ still satisfies the K-G eq.

$\Rightarrow$ ψ still has both positive- and NEGATIVE-energy solutions

$$j^\mu = (\rho, \vec{j}) = \psi^+ \gamma^0 \gamma^\mu \psi$$

$$\rho = \psi^+ (\gamma^0)^2 \psi = \psi^+\psi$$ - POSITIVE-DEFINITE!

Dirac eq.

- describes the electron (charged, spin 1/2 particle) and in the nonrelativistic limit one obtains the correct gyromagnetic ratio for electron

$$i\frac{\partial \varphi}{\partial t} = \left[\frac{\hat{\vec{p}}^2}{2m} - \frac{e}{2m}(\vec{L} + \underset{\substack{\uparrow \\ g=2!}}{2}\vec{S})\cdot\vec{B}\right]\varphi, \quad \vec{S} = \frac{\vec{\sigma}}{2}$$

σ_i, $i=1,2,3$ Pauli's matrices

- Since spin 1/2 particles obey Pauli's exclusion principle

$\Downarrow$

HOLE THEORY - to explain the negative-energy solutions

Vacuum state : all negative-energy levels are already occupied

1931 - holes = anti-electrons (mass and spin identical to the electron's, but opposite electric charge)

1932 - C. Anderson - experimental discovery of positrons

1928 Bohr
- "Bohr - Heisenberg commutation relations for the bosonic field

Jordan, Pauli, Wigner
- general method for the quantization of physical systems (including Pauli's exclusion principle, proposed in 1925)

1929 Heisenberg and Pauli
- the method of canonical quantization of fields ("operator formulation")

1934 Pauli and Weisskopf
- quantization of the scalar field

1936-1940 Pauli
- the spin-statistics relation (theorem)

1929-1930 Pauli, Heisenberg
- the problem of infinities in QFT...

1948 Feynman (b. 1918)
- path-integral approach
- "Feynman rules"

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