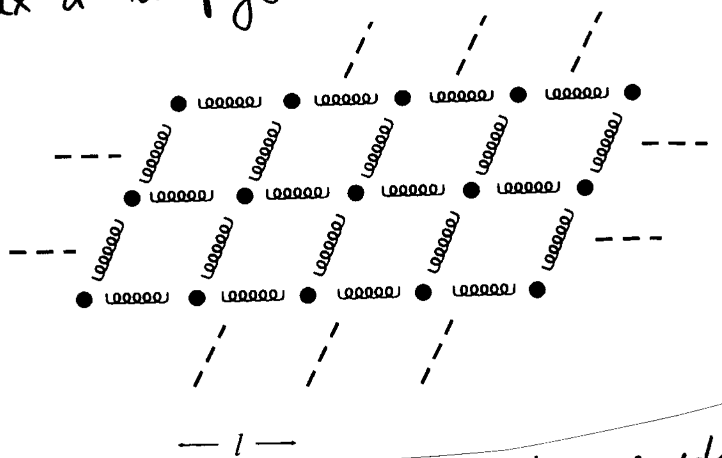
## Transition to continuum. The concept of field

Consider a 2-dimensional lattice of point masses connected to each other by springs. Denote by $q_a(t)$ the displacement of the point mass "a" from the position of equilibrium and denote by $l$ the spacing of the lattice. The index "a" simply labels the masses on the lattice.




The Lagrangean of the system is:

$$L = \frac{1}{2}\left(\sum_a m_a \dot{q}_a^2 - \sum_{a,b} k_{ab}\, q_a q_b - \sum_{a,b,c} g_{abc}\, q_a q_b q_c - \dots\right)$$

Take the continuum limit, $l \to 0$ (suppose we are interested in phenomena on length scales much greater than the lattice spacing $l$).

In this limit, we can replace the label "a" by a 2-dimensional position vector $\vec{x}$

$$q_a(t) \to q(t,\vec{x}) \equiv \varphi(t,\vec{x})$$

The function $\varphi(t,\vec{x})$ is called a <u>field</u>.

- Replace $\sum_a$ by $\frac{1}{l^2}\int d^2x$

- Denote the mass per unit area

$$\sigma = \frac{m_a}{\ell^2}$$

(assume all $m_a$'s to be equal, so that $\sigma$ is not a function of position)

- The kinetic energy $\sum_a \frac{1}{2} m_a \dot{q}_a^2$ becomes then

$$\int d^2x \ \frac{1}{2}\sigma\left(\frac{\partial \varphi}{\partial t}\right)^2$$

- Take the first term of the potential, $\sum_{a,b} \frac{1}{2} k_{ab} q_a q_b$.
- Write $-2q_a q_b = (q_a - q_b)^2 - q_a^2 - q_b^2$ and assume that $k_{ab}$ connect only nearest neighbours in the lattice
- For nearest-neighbours pairs, in the continuum limit

$$\left(\frac{q_a - q_b}{\ell}\right)^2 \longrightarrow \left(\frac{\partial \varphi}{\partial x}\right)^2$$

with the derivative taken in the direction that joins the lattice sites "a" and "b".

Then the Lagrangean becomes:

$$L = \int d^2x \ \mathcal{L}(\varphi) =$$

$$= \int d^2x \ \frac{1}{2}\left\{\sigma\left(\frac{\partial \varphi}{\partial t}\right)^2 - \rho\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] - \right.$$

$$\left. - \tau \varphi^2 - \xi \varphi^3 - \dots \right\}$$

$$= \int d^2x \ \frac{1}{2}\left\{\left(\frac{\partial \varphi}{\partial t}\right)^2 - \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] - \tau' \varphi^2 - \xi' \varphi^3 - \dots\right.$$

(where we took $\rho/\sigma = c^2$ and then set $c = 1$)

Remark 1 : In field theory, the dynamical variable is the field $\varphi(t,\vec{x})$ and not the coordinate $\vec{x}$. Since $\vec{x}$ appearing in $\varphi(t,\vec{x})$ is a mere label (just like $a$ in $q_a(t)$ for usual particle systems) and since $\vec{x}$ can take any value in $R^2$ (for the present 2-dimensional case), a field is a system with an infinite number of degrees of freedom (d.o.f.) In other words, when passing to the continuum limit, we have the correspondences:

$$q \rightarrow \varphi$$
$$a \rightarrow \vec{x}$$
$$q_a(t) \rightarrow \varphi(t,\vec{x}) = \varphi(x) \;; \quad (x = (t,\vec{x}))$$
$$\sum_a \rightarrow \int d^D x \quad (\text{for } D\text{-dim. space})$$

This crucial distinction in the role of the coordinate $\vec{x}$ will be better seen at quantum level: in quantum mechanics (finite no. of d.o.f.) the coordinate $\hat{x}_i$ and the conjugate momentum $\hat{p}_j$ are operators, satisfying the commutation relation

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \quad (\hbar = 1)$$

In quantum field theory (infinite no. of d.o.f.), the coordinates $x$ are c-numbers and $\varphi(x)$ and the canonically conjugate momentum $\pi(x)$ are operators, depending on the coordinates $x$ as parameters and satisfying the commutation relations (at equal times)

$$[\varphi(\vec{x},t), \pi(\vec{x}',t)] = i\delta(\vec{x}-\vec{x}')$$

Remark 2: The role of this example was to emphasize the fact that a field is a system with an infinite no. of degrees of freedom, depending on the coordinates as parameters. Of course nobody believes that the fields observed in Nature (e.g. a meson or a photon field) are actually constructed of point masses tied together with springs!

The modern view is to start with the desired symmetry, e.g. Poincaré invariance if we want to do particle physics, decide on the fields we want by specifying their transformation properties under that symmetry and then write down the invariant action (or Lagrangean density) involving no more than two time derivatives (because we do not know how to quantize actions with more than two time derivatives).

# Lagrangean and Hamiltonian formalism for fields

- In order to extend the commutation relation of QM

$$[\hat{q}_i(t), \hat{p}_j(t)] = i\delta_{ij}$$

to systems with infinite no. of degrees of freedom, we have to work with Heisenberg operators and use the Hamiltonian formalism, in which time has a privileged role, compared to the space coordinates. Due to this privileged role of time, the Hamiltonian formalism is not manifestly relativistic (Poincaré) covariant. However, one can construct manifestly Poincaré invariant Lagrangean densities for relativistic fields and then go to the Hamiltonian formalism

$$\left. \begin{array}{l} S = \int_{t_0}^{t} dt\, L = \int_{t_0}^{t} dt \int_V d^3x\, \mathcal{L}(x) = \int_\Omega d^4x\, \mathcal{L}(x) \\ S - \text{relativistically invariant action} \end{array} \right] \Rightarrow$$

$\Rightarrow \mathcal{L}(x)$ - relativistically invariant Lagrangean density

| Point-like particle mechanics | | Relativistic classical fields |
|---|---|---|
| $q_a$ - generalized coordinate, dynamical var. | $\longrightarrow$ | $\varphi$ - the field - generalized coordinate |
| $t$ - time variable | $\longrightarrow$ | $x^\mu \equiv x$ space-time coordinates |
| $L(t) = L(q_a(t), \dot{q}_a(t))$ | $\longrightarrow$ | $L(t) = \int_V d^3x\, \mathcal{L}(x)$, $\boxed{\mathcal{L}(x) = \mathcal{L}(\varphi(x), \partial_\mu \varphi(x))}$ |

- Furthermore we assume:
  - Lagrangean density is a real function
  - Lagr. density contains fields and their derivatives taken at one point only $\Rightarrow$ LOCAL field theories.
  - Lagr. density depends only on the fields $\varphi_a(x)$ and their derivatives $\partial_\mu \varphi_a(x)$. We also restrict ourselves to theories with two derivatives at most.
  - Lagr. density does not depend EXPLICITLY on the space-time coord. $x^\mu$ (explicit dependence on $x^\mu$ would spoil the translational invariance)
- Principle of least action $\Rightarrow$ field equations of motion (Euler-Lagrange equations).

$$0 = \delta S = \int_\Omega d^4x \, \delta \mathcal{L}(\varphi_a(x), \partial_\mu \varphi_a(x))$$

$$= \int_\Omega d^4x \left[ \frac{\partial \mathcal{L}}{\partial \varphi_a} \delta \varphi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta \partial_\mu \varphi_a \right]$$

$$= \int_\Omega d^4x \left[ \frac{\partial \mathcal{L}}{\partial \varphi_a} \delta \varphi_a - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta \varphi_a \right) \right]$$

$$= \int_\Omega d^4x \left[ \frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) \right] \delta \varphi_a + \underbrace{\int_{\partial\Omega} dS_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta \varphi_a \right)}_{0 \ \text{(since } \delta\varphi_a|_{\partial\Omega} = 0)}$$

$$= \int_\Omega d^4x \left[ \frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) \right] \delta \varphi_a$$

Since $\delta \varphi_a$ are arbitrary variations, it follows that

$$\boxed{\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} = 0}$$

Euler-Lagrange equations for the fields $\varphi_a$

- Hamiltonian formalism

Point-like particle mechanics | Relativistic classical field

canonically conjugated momentum

– to the coordinate $q_a$

$$p_a(t) = \frac{\partial L}{\partial \dot{q}_a(t)}$$

– to the field $\varphi_a$

$$\pi_a(x) = \frac{\partial \mathcal{L}(x)}{\partial(\partial_0 \varphi_a(x))}$$

Hamiltonian

$$H(q_a(t), p_a(t)) = \sum_a p_a(t)\dot{q}_a(t) - L(q_a, \dot{q}_a)$$

$$H = \int d^3x\, \mathcal{H}(x)$$

$$\mathcal{H}(x) = \sum_a \pi_a(x)\partial_0\varphi_a(x) - \mathcal{L}(\varphi_a(x), \partial_\mu \varphi_a(x))$$

Hamilton equations

$$\dot{p}_a(t) = -\frac{dH}{dq_a}$$

$$\dot{q}_a(t) = \frac{dH}{dp_a}$$

$$\partial_0 \pi_a(x) = -\frac{\partial \mathcal{H}}{\partial \varphi_a(x)} + \frac{\partial}{\partial x_i}\frac{\partial \mathcal{H}}{\partial(\partial_i \varphi_a)}$$

$$\partial_0 \varphi_a(x) = \frac{\partial \mathcal{H}}{\partial \pi_a(x)}$$

- Examples of relativistically invariant Lagrangean densities:

– free case

i) real scalar field $\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 \Longrightarrow (\Box + m^2)\varphi = 0$

ii) complex scalar field $\mathcal{L} = \partial^\mu\varphi^*\partial_\mu\varphi - m^2\varphi^*\varphi \Longrightarrow \begin{cases}(\Box + m^2)\varphi = 0 \\ (\Box + m^2)\varphi^* = 0\end{cases}$

iii) spinor field $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \Longrightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0$

iv) electromagn. field $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \longrightarrow \partial_\mu F^{\mu\nu} = 0$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

– systems with interaction

i) $\lambda\varphi^4$ – theory $\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi - \frac{\lambda}{4!}(\varphi^*\varphi)^2$

ii) electrodynamics (int. of electrons with electromagn. field) $\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu$.

## Dimensional analysis in natural units

The system of units best suited for particle physics is the system of natural units. In natural units, one takes

$M$ - mass
$A$ - action
$V$ - velocity

as fundamental dimensions and choses

$\hbar$ as unit of action
$c$ as unit of velocity

i.e. $\hbar = c = 1$

As a result, in natural units all quantities have the dimensions of a power of mass, $M^n$.

| Quantity | Dimension in natural units $M^n$ |
|---|---|
| Action | $M^0$ |
| Velocity | $M^0$ |
| Mass | $M$ |
| Length | $M^{-1}$ |
| Time | $M^{-1}$ |
| Momentum | $M$ |
| Energy | $M$ |
| Lagrangean or Hamiltonian density | $M^4$ |
| Klein-Gordon field $\phi(x)$ | $M^1$ |
| Electromagnetic field $A^\mu(x)$ | $M^1$ |
| Dirac field $\psi(x)$ | $M^{3/2}$ |

(The dimensions of the fields $\phi(x)$, $A^\mu(x)$, $\psi(x)$ can be obtained from the corresponding free Lagrangean densities)

## Relativistic notation

Natural units $c = \hbar = 1$

- $x^\mu$ - space-time four-vector $\mu = 0, 1, 2, 3$

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \vec{x})$$

$x^0 = t$ - time component

$x^j$, $j = 1, 2, 3$ - space coordinates

- $$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathrm{diag}(1, -1, -1, -1)$$ - metric tensor

$$x_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\nu \equiv \eta_{\mu\nu} x^\nu$$ - summation convention: Repeated indices are always summed over

Note: A <u>summation index</u> cannot appear in a product more than <u>twice</u>!

$$\eta^{\lambda\mu}\eta_{\mu\rho} = \delta^\rho_\mu \qquad (\delta^\rho_\mu \text{ - Kronecker delta})$$

$$\eta^{\mu\nu} = \eta_{\mu\nu}$$

- A Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad , \Lambda \text{ - real matrix}$$

leaves the 4-dimensional interval

$$x^\mu x_\mu = (x^0)^2 - x^i x^i = (x^0)^2 - (\vec{x})^2$$

<u>invariant</u>, i.e $x'^\mu x'_\mu = x^\mu x_\mu$ is a <u>scalar</u> quantity

Hence $\Lambda^{\lambda\mu}\Lambda_{\lambda\nu} = \delta^\mu_\nu$

- A four-component object $V_\mu$, transforming like $x^\mu$ under Lorentz transformation (hence with $V^\mu V_\mu$ a scalar) is a four-vector

Example : energy-momentum vector

$$p^\mu = (E, \vec{p})$$

- Scalar product

$$ab = a^\mu b_\mu = a_\mu b^\mu = \eta_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \vec{a}\cdot\vec{b}$$

The <u>scalar product</u> $ab$ is an <u>invariant</u> under Lorentz transformations

- $\partial_\mu = \frac{\partial}{\partial x^\mu}$ – the 4-dimensional generalization of the gradient $\nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$

$\partial^\mu$ transforms like a 4-vector!

$\phi(x)$ – scalar function

$$\left.\begin{aligned} \frac{\partial \phi(x)}{\partial x^\mu} &\equiv \partial^\mu \phi(x) \equiv \phi^{(x)}_{,\mu} \\ \frac{\partial \phi(x)}{\partial x_\mu} &\equiv \partial_\mu \phi(x) \equiv \phi_{,(x)}^{\mu} \end{aligned}\right\} \text{vector functions!}$$

- d'Alembertian operator $\Box$ is a scalar

$$\Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^i \partial x^i} = \frac{\partial^2}{\partial t^2} - \Delta$$

$$(\partial_\mu = (\partial_t, \nabla))$$