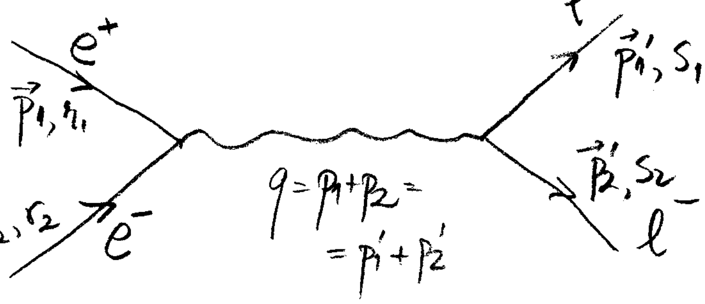
Lepton pair production in e^+e^- collisions

$$e^+e^- \to \ell^+\ell^-$$



$$\mathcal{M} = \bar{u}_{s_2}(\vec{p}_2')(-ie\gamma^\mu)v_{s_1}(\vec{p}_1')\,\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2}\,\bar{v}_{r_1}(\vec{p}_1)(-ie\gamma^\nu)u_{r_2}(\vec{p}_2)$$

Unpolarized cross-section:
- average over initial particles' spin polarizations
- sum over final particles' spin polarizations

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{r_1}\frac{1}{2}\sum_{r_2}\sum_{s_1}\sum_{s_2}\left(\mathcal{M}^*\mathcal{M}\right)$$

$$\mathcal{M}^* = \bar{v}_{s_1}(\vec{p}_1')(ie\gamma^\alpha)u_{s_2}(\vec{p}_2')\,\frac{i\eta_{\alpha\beta}}{(p_1+p_2)^2}\,\bar{u}_{r_2}(\vec{p}_2)(ie\gamma^\beta)v_{r_1}(\vec{p}_1)$$

since $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

$$\left(\text{e.g. } (\bar{u}\gamma^\mu v)^* = (u^\dagger\gamma^0\gamma^\mu v)^\dagger = (u^\dagger \underbrace{\gamma^0\gamma^\mu\gamma^0}_{\gamma^{\mu\dagger}}\gamma^0 v)^\dagger\right.$$

$$= (u^\dagger\gamma^{\mu\dagger}\gamma^0 v)^\dagger = v^\dagger\gamma^0\gamma^\mu u = \bar{v}\gamma^\mu u$$

Then

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4(p_1+p_2)^4}\sum_{s_1,s_2,r_1,r_2}\left[\bar{v}_{s_1}(\vec{p}_1')\gamma^\alpha u_{s_2}(\vec{p}_2')\right]\left[\bar{u}_{r_2}(\vec{p}_2)\gamma_\alpha v_{r_1}(\vec{p}_1)\right]$$

$$\times\left[\bar{u}_{s_2}(\vec{p}_2')\gamma^\mu v_{s_1}(\vec{p}_1')\right]\left[\bar{v}_{r_1}(\vec{p}_1)\gamma_\mu u_{r_2}(\vec{p}_2)\right]$$

$$= \frac{e^4}{4(p_1+p_2)^4}\sum_{s_1,s_2}\left[\bar{v}_{s_1}(\vec{p}_1')\gamma^\alpha u_{s_2}(\vec{p}_2')\right]\left[\bar{u}_{s_2}(\vec{p}_2')\gamma^\mu v_{s_1}(\vec{p}_1')\right]$$

$$\times\sum_{r_1,r_2}\left[\bar{u}_{r_2}(\vec{p}_2)\gamma_\alpha v_{r_1}(\vec{p}_1)\right]\left[\bar{v}_{r_1}(\vec{p}_1)\gamma_\mu u_{r_2}(\vec{p}_2)\right]$$

$$\sum_{s_1} v_{s_1}(\vec{p}_1{}') \bar{v}_{s_1}(\vec{p}_1{}') = \not{p}_1' - m_\ell$$

$$\sum_{s_2} u_{s_2}(\vec{p}_2{}') \bar{u}_{s_2}(\vec{p}_2{}') = \not{p}_2' + m_\ell$$

$$\sum_{r_1} v_{r_1}(\vec{p}_1) \bar{v}_{r_1}(\vec{p}_1) = \not{p}_1 - m_e$$

$$\sum_{r_2} u_{r_2}(\vec{p}_2) \bar{u}_{r_2}(\vec{p}_2) = \not{p}_2 + m_e$$

$$\Rightarrow \overline{|\mathcal{M}|}^2 = \frac{e^4}{4(p_1+p_2)^4} \mathrm{Tr}\left[(\not{p}_1' - m_\ell)\gamma^\alpha(\not{p}_2' + m_\ell)\gamma^\mu\right] \times \mathrm{Tr}\left[(\not{p}_2 + m_e)\gamma_\alpha(\not{p}_1 - m_e)\gamma_\mu\right]$$

$$\mathrm{Tr}\left[(\not{p}_1' - m_\ell)\gamma^\alpha(\not{p}_2' + m_\ell)\gamma^\mu\right] = \mathrm{Tr}(\not{p}_1'\gamma^\alpha\not{p}_2'\gamma^\mu) - m_\ell^2\,\mathrm{Tr}(\gamma^\alpha\gamma^\mu)$$

- use

$$\mathrm{Tr}(\gamma^\beta\gamma^\alpha\gamma^\nu\gamma^\mu) = 4(\eta^{\beta\alpha}\eta^{\nu\mu} - \eta^{\beta\nu}\eta^{\alpha\mu} + \eta^{\beta\mu}\eta^{\alpha\nu})$$

$$\mathrm{Tr}(\gamma^\alpha\gamma^\mu) = 4\eta^{\alpha\mu}$$

$$\mathrm{Tr}(\not{p}_1'\gamma^\alpha\not{p}_2'\gamma^\mu) = p'_{1\beta}\, p'_{2\nu}\, \mathrm{Tr}(\gamma^\beta\gamma^\alpha\gamma^\nu\gamma^\mu) =$$

$$= p'_{1\beta}\, p'_{2\nu}\, 4(\eta^{\beta\alpha}\eta^{\nu\mu} - \eta^{\beta\nu}\eta^{\alpha\mu} + \eta^{\beta\mu}\eta^{\alpha\nu})$$

$$= 4(p_1'^{\alpha} p_2'^{\mu} - p_1'^{\nu} p'_{2\nu}\eta^{\alpha\mu} + p_1'^{\mu} p_2'^{\alpha})$$

$$\mathrm{Tr}\left[(\not{p}_1' - m_\ell)\gamma^\alpha(\not{p}_2' + m_\ell)\gamma^\mu\right] = 4\left(p_1'^{\alpha} p_2'^{\mu} + p_1'^{\mu} p_2'^{\alpha} - (m_\ell^2 + p_1' p_2')\eta^{\alpha\mu}\right)$$

Similarly,

$$\mathrm{Tr}\left[(\not{p}_2 + m_e)\gamma_\alpha(\not{p}_1 - m_e)\gamma_\mu\right] = 4\left(p_{1\alpha} p_{2\mu} + p_{1\mu} p_{2\alpha} - (m_e^2 + p_1 p_2)\eta_{\alpha\mu}\right)$$

Then

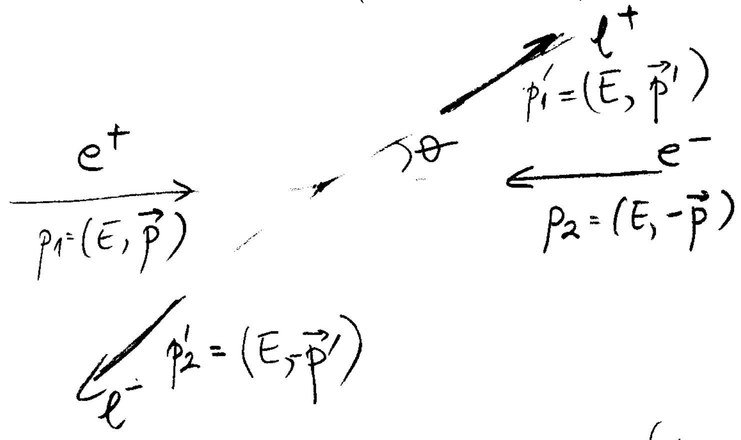
$$\overline{|\mathcal{M}|^2} = \frac{e^4}{(p_1+p_2)^4} 4\Big[2(p_1 p_1')(p_2 p_2') + 2(p_1' p_2)(p_2' p_1) -$$

$$-2(p_1' p_2')(p_1 p_2) - 2(p_1 p_2)(p_1' p_2') + 4(p_1 p_2)(p_1' p_2')$$

$$+ 2m_e^2 (p_1' p_2') + 2m_\ell^2 (p_1 p_2) + 4 m_e^2 m_\ell^2\Big]$$

COM frame

$$\left(\frac{d\sigma}{d\Omega}\right)_{com} = \frac{1}{64\pi^2 (E_1+E_2)^2} \frac{|\vec{p}_1'|}{|\vec{p}_1|} \overline{|\mathcal{M}|^2}$$



$$p_1 + p_2 = (2E, 0) \Rightarrow (p_1+p_2)^2 = 4E^2$$

$$p_1 + p_2 = p_1' + p_2'$$

$$p_1 p_1' = E^2 - |\vec{p}||\vec{p}'| \cos\theta$$

$$p_1 p_2' = E^2 - |\vec{p}||\vec{p}'| \cos(\pi - \theta) = E^2 + |\vec{p}||\vec{p}'| \cos\theta$$

$$p_1 p_2 = E^2 + |\vec{p}|^2$$

$$p_1' p_2' = E^2 + |\vec{p}'|^2$$

Since $m_e \ll m_\ell$, we can neglect the mass of the electron

$$p_1^2 = E^2 - |\vec{p}|^2 = \underset{\text{negl.}}{m_e^2} \Rightarrow |\vec{p}|^2 = E^2$$

$$|\overline{M}|^2 = \frac{e^4}{4E^4}\Big[2\left(E^2 - |\vec{p}||\vec{p}\,'|\cos\theta\right)^2 + 2\left(E^2 + |\vec{p}||\vec{p}\,'|\cos\theta\right)^2$$

$$+ 2m_\ell^2\left(E^2 + |\vec{p}|^2\right)\Big] =$$

$$= \frac{e^4}{2E^4}\left[2E^4 + 2|\vec{p}|^2|\vec{p}\,'|^2\cos^2\theta + 2m_\ell^2 \cdot 2E^2\right]$$

$$= \frac{e^4}{E^4}\left[E^4 + E^2|\vec{p}\,'|^2\cos^2\theta + m_\ell^2 E^2\right]$$

$$= \frac{e^4}{E^2}\left[E^2 + |\vec{p}\,'|^2\cos^2\theta + m_\ell^2\right]$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{COM} = \frac{e^4}{16\cdot 16\pi^2 E^4}\,\frac{|\vec{p}\,'|}{E}\left[E^2 + |\vec{p}\,'|^2\cos^2\theta + m_\ell^2\right]$$

$$= \frac{\alpha^2}{16E^4}\,\frac{|\vec{p}\,'|}{E}\left[E^2 + |\vec{p}\,'|^2\cos^2\theta + m_\ell^2\right]$$

$$\sigma = \int d\sigma = \int\left(\frac{d\sigma}{d\Omega}\right)_{COM} d\Omega = \int_0^{2\pi} d\varphi \int_0^1 d\cos\theta\left(\frac{d\sigma}{d\Omega}\right)_{COM}$$

$$\int_0^{2\pi} d\varphi = 2\pi$$

$$\int_{-1}^1 d\cos\theta = 2\,, \quad \int_{-1}^1 \cos^2\theta\, d\cos\theta = \frac{\cos^3\theta}{3}\Big|_0^1 = \frac{2}{3}$$

$$\sigma = \frac{\pi\alpha^2}{8E^4}\,\frac{|\vec{p}\,'|}{E}\left[E^2 + m_\ell^2 + \frac{1}{3}|\vec{p}\,'|^2\right]$$