

Due at 8.15 on Thursday 24 November 2011

1. Show that the singlet state $(|+\rangle|-\rangle - |-\rangle|+\rangle)/\sqrt{2}$ is also an eigenstate of the operators $S_x = s_{1x} + s_{2x}$ and $S_y = s_{1y} + s_{2y}$. What is the eigenvalue? Are the triplet states also eigenstates of S_x as well as S_z ? Why not, what is the difference in that case?
2. Compute the eigenstates of σ_x in the base of σ_z eigenstates. Does $(|+\rangle_z + |-\rangle_z)/\sqrt{2}$ describe an unpolarized beam, as it has equal probabilities of $s_z = \pm\hbar/2$?

Consider an electron in a constant magnetic field directed along the z -axis. The magnetic potential is then $V = -\boldsymbol{\mu} \cdot \mathbf{B}$, where the magnetic dipole moment equals $\boldsymbol{\mu} = 2\mu_B \mathbf{S}/\hbar$, with μ_B so called Bohr magneton. Using this as the Hamiltonian operator write down the time development of the state, assuming that at time $t = 0$ the system is in the above state $|+\rangle_x$. Compute the time development of the expectation value of spin operator. (You can ignore all other contributions to energy.)

3. Let the Hamiltonian operator of two spin- $\frac{1}{2}$ particles with fixed positions be

$$H = \frac{a}{\hbar}(S_{1z} + S_{2z}) + \frac{4b}{\hbar^2} \mathbf{S}_1 \cdot \mathbf{S}_2,$$

where a, b are real and positive. Calculate the eigenvalues of energy and exhibit the eigenstates both in the coupled basis $|\frac{1}{2} \frac{1}{2} S m\rangle$ (what are the possible values of the total spin S ?) and in the uncoupled basis $|\frac{1}{2} \frac{1}{2} m_1 m_2\rangle$.

4. Consider a hydrogen-like atom in the first excited states $n = 2, \ell = 1$. Compute the probability density of the electron as a function of the angles θ and φ in states, whose *total* angular momentum has z -component $+\hbar/2$. (You can find the tabulated Clebsch–Gordan coefficients on course home page.)

Also compute (in perturbation theory) the energy difference between states $j = 3/2$ and $j = 1/2$ coming from the spin–orbit coupling

$$V_{\text{SO}}(\mathbf{r}) = \frac{Ze^2}{4\pi\epsilon_0} \frac{\mathbf{S} \cdot \mathbf{L}}{2m_e^2 c^2 r^3}.$$

Useful fact: wave functions of a hydrogen-like atom satisfy

$$\langle n\ell | \frac{1}{r^3} | n\ell \rangle = \left(\frac{Z}{a_0^3} \right) \frac{2}{n^3 \ell(\ell+1)(2\ell+1)}.$$