KVANTTIMEKANIIKKA I Autumn 2011 Exercise 11

Due at 8.15 on Thursday 1st December 2011

1. As you know, harmonic oscillator eigenstates can be derived starting from the commutation relations of raising and lowering operators a and $a^\dagger$:

$$[a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0, \quad H_b = \frac{\hbar\omega_b}{2}(a^\dagger a + aa^\dagger)$$

A fermionic (anticommuting) oscillator can be defined the same way using *anticommuting* operators $b, b^\dagger$ and the anticommutator defined by $\{A, B\} \equiv AB+BA$ for arbitrary A, B (note that the bosonic Hamiltonian above is just $\{a^\dagger, a\}$) :

$$\{b, b^\dagger\} = 1, \quad \{b, b\} = \{b^\dagger, b^\dagger\} = 0, \quad H_f = \frac{\hbar\omega_f}{2}[b^\dagger, b]$$

   (a) Find the energy eigenvalues and eigenstates of the fermionic oscillator (very simple system).

   (b) If we couple the two kinds of oscillators together as $H = H_b + H_f$ with $\omega_b = \omega_f \equiv \omega$ (and a's commute with b's), show that the zero point (ground state) energy vanishes. Also show that the total Hamiltonian can be written as $H = \hbar\omega\{Q, Q^\dagger\}$ with $Q = b^\dagger a$.

2. Two different particles of masses m_1 and m_2 move in one dimension and are not subject to any external forces. The potential energy for the interaction between the particles is given by

$$V = 0 \quad (|x_{12}| \le a); \qquad V = \infty \quad (|x_{12}| > a),$$

where $x_{12} \equiv x_1 - x_2$ is the particle separation.

Obtain expressions for the energy eigenvalues and eigenfunctions of this system if its total momentum is P.

3. What are the energy eigenvalues and eigenfunctions in problem 2 if the two particles have the same mass and are: (i) identical spin-zero bosons and (ii) identical spin-half fermions.

4. On the lecture we derived a formula for the density of states for given energy. Nothing in the derivation depended on the direction of $\mathbf{p}$, so the density of spatial states per energy and solid angle reads (divide by 4π and drop the spin degeneracy for the moment)

$$\rho(E)\mathrm{d}E\mathrm{d}\Omega = \frac{L^3}{16\pi^3}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E} = \left(\frac{L}{2\pi}\right)^3 \frac{mk}{\hbar^2}\,\mathrm{d}E\mathrm{d}\Omega.$$

Derive an analogous formula for the density of states of a free particle in *two* dimensions, *i.e.* $\rho(E)\mathrm{d}E\mathrm{d}\phi$.

5. A tritium nucleus decays into helium through ${}^3\mathrm{H} \to {}^3\mathrm{He} + e^- + \bar{\nu}_e$. From the point of view of a bound state electron the change in the potential (nuclear charge $Z = 1 \to 2$) is sudden, and the extra electron flies away very fast. If the hydrogen electron was initially in the ground state, what is the probability of finding the resulting ${}^3\mathrm{He}^+$ ion in the ground state?