KVANTTIMEKANIIKKA I Autumn 2011 Exercise 5

Due at 8.15 on Thursday 13 October 2011

1. Assume a hydrogen atom is in the superposition state
$$\Psi(\mathbf{r}) = N\left[\psi_{100}(\mathbf{r}) + \psi_{210}(\mathbf{r}) - \sqrt{3}\psi_{211}(\mathbf{r})\right],$$
where $\psi_{n\ell m}(\mathbf{r})$ are hydrogen energy eigenstates. What are the possible values when measuring E, $\mathbf{L}^2$ and L_z, and what are their probabilities? Compute the expectation values $\langle E\rangle$, $\langle \mathbf{L}^2\rangle$ and $\langle L_z\rangle$.

2. At time $t = 0$ the state of a hydrogen atom is a superposition of two s-wave ($\ell = 0$) states
$$\Psi(\mathbf{r}) = N\left[\psi_{100}(\mathbf{r}) + \psi_{200}(\mathbf{r})\right].$$
Compute $\langle r\rangle$ and its time dependence. What would happen to the time dependence if the latter term had a different angular momentum, say $\ell = 1$?

3. Let $\{|a_1\rangle, |a_2\rangle\}$ be an orthonormal basis ("a-representation") in a two-dimensional Hilbert space, and define the ket states
$$|b_1\rangle = \frac{1}{\sqrt{3}}(|a_1\rangle + i\sqrt{2}|a_2\rangle), \qquad |b_2\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|a_1\rangle - i|a_2\rangle).$$
Show that $\{|b_1\rangle, |b_2\rangle\}$ is also an orthonormal basis ("b-representation") and represent vectors $|a_1\rangle$ and $|a_2\rangle$ in this basis. Write down the matrix transforming from one base to another. Is it unitary?

4. Define the orbital angular momentum (operator) via $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$. Compute the commutators between different components ($\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$) of $\hat{\mathbf{L}}$.

5. **Two-body oscillator** Consider a system of two particles of mass m bound together by a harmonic potential,
$$H = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 + \frac{1}{2}\kappa(\mathbf{r}_1 - \mathbf{r}_2)^2.$$
Write down the ground state energy and wave function. (Assume that the center of mass of the system is at rest.)

   Hint: you can solve this in either cartesian or spherical coordinates. If you have time and energy, do both and convince yourself that you get the same result.