KVANTTIMEKANIIKKA I Autumn 2011 Exercise 6

Due at 8.15 on Thursday 20 October 2011

1. Two Hermitean operators $\hat{A}$ and $\hat{B}$ satisfy the commutation relation $[\hat{A}, \hat{B}] = i\mathbb{1}$. (Think of our Postulate 3.) Show using induction that
$$[\hat{A}, \hat{B}^n] = in\hat{B}^{n-1} \qquad \text{and} \qquad [\hat{B}, \hat{A}^n] = -in\hat{A}^{n-1}.$$
If $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}_x$ (in one-dimensional space), what does this imply for commutators of analytic functions of $\hat{x}$ and $\hat{p}_x$ with $\hat{x}$ and $\hat{p}_x$? How about with the kinetic energy $\hat{E}_{\text{kin}} = \hat{p}_x^2/2m$?

2. Let λ be some parameter appearing in the Hamiltonian. Prove the following theorem, useful for computing expectation values:
$$\frac{\partial E_n}{\partial \lambda} = \langle \psi_n | \frac{\partial H}{\partial \lambda} | \psi_n \rangle,$$
where E_n and $|\psi_n\rangle$ are the eigenvalue and eigenstate of some bound state of H.
Using this, compute $\langle 1/r \rangle$ in a hydrogen bound state (choose e.g. $\lambda = e^2$).
(For fun, you can also try computing $\langle 1/r^2 \rangle$, which is a bit trickier. This is not required nor will it give extra points, however.)

3. A two-state system is in a mixed state, with each of the states
$$|\Psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |\Psi_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\Psi_3\rangle = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
occurring with equal weights. (Note: these states are not all orthogonal.)
Compute the density matrix and the probabilities for the system being found in the state
$$\text{a)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{b)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{c)} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

4. Solve the following algebraically (*i.e.*without wave functions) for a one-dimensional harmonic oscillator:
   (a) Determine the superposition of states $|0\rangle$ and $|1\rangle$ which gives the largest expectation value $\langle x \rangle$.
   (b) If at time $t = 0$ the oscillator is in the state of problem (a), what is the state at some later time $t > 0$? Compute also $\langle x \rangle$ as a function of time.

5. At thermal equilibrium, the density operator of a system is
$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}},$$
where Z is a constant fixed by the normalization $\text{Tr}\,\hat{\rho} = 1$, and $\beta = 1/T$ is the inverse of temperature (measured in energy units, $k_B = 1$). For a harmonic oscillator at temperature T, compute
   (a) The constant (*partition function*) $Z(\beta)$.
   (b) The expectation value of energy, $\langle E \rangle$.