Session of Tuesday 21 February at 16-18 in aud A315

1. (a) The SO(2) matrix $R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ specifies a rotation of $\boldsymbol{x} = (x_1, x_2)^T$. Give the corresponding generator(s) of rotations.

   (b) Show that $R(\phi)$ is equivalent to a reducible representation of the group U(1), by finding a fixed $V = (V^\dagger)^{-1}$ such that $U_R(\phi) \equiv V\, R(\phi)\, V^\dagger = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$ for all ϕ.
   Hint: You may use $W\sigma_2 = \sigma_3 W$ for $W = \frac{1}{2}(1 + i\boldsymbol{s}\cdot\boldsymbol{\sigma})$ with $\boldsymbol{s} = (-1, 1, 1)$.

   (c) For $\boldsymbol{x} = (x_1, x_2)$ define $X = \boldsymbol{x}\cdot\boldsymbol{\sigma}$ and show that $X' = U_R(\phi)\, X\, U_R^\dagger(\phi)$ specifies a rotation of $\boldsymbol{x}$. Find the rotation angle.

2. The generators J^a of a (compact semi-simple) Lie group satisfy the Lie algebra $\left[J^a, J^b\right] = iC_{abc}J^c$, with structure constants C_{abc} that are fully antisymmetric: $C_{abc} = -C_{bac} = C_{bca}\ldots$ In the *adjoint* matrix representation the generators are given by $(J^b)_{ac} = iC_{abc}$.

   (a) Use the Jacobi identity to show that the generators in the adjoint representation satisfy the Lie algebra.

   (b) The SU(2) structure constants are $C_{abc} = \epsilon_{abc}$ ($a, b, c = 1, 2, 3$ with $\epsilon_{123} = 1$). Give the generator matrices in the adjoint representation and the eigenvectors of J_3.

3. Consider the 2×2 matrix defined by $U = (a_0 + i\vec{\sigma}\cdot\vec{a})/(a_0 - i\vec{\sigma}\cdot\vec{a})$, where a_0 is a real number and $\vec{a}$ is a three-dimensional vector with real components. In general, an $SU(2)$ matrix defines a rotation in three dimensions through the relation $\vec{\sigma}\cdot\vec{x}' = U\vec{\sigma}\cdot\vec{x}\, U^\dagger$.

   (a) Prove that $U \in SU(2)$, *i.e.*, that U is unitary and unimodular ($\det U = 1$).

   (b) Determine the axis of rotation corresponding to U in terms of $a_0, \vec{a}$.

   (c) Show that the angle ϕ of the rotation satisfies $\sin(\phi/2) = 2a_0|\vec{a}|/(a_0^2 + \vec{a}^2)$ (the sign depends on whether the rotation is considered to be active or passive).
   Hint: You may choose U to be a rotation around the z-axis here.

4. Consider a sequence of Euler rotations represented by

$$\begin{aligned} \mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) &= \exp(-i\sigma_3\alpha/2)\,\exp(-i\sigma_2\beta/2)\,\exp(-i\sigma_3\gamma/2) \\ &= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix} \end{aligned}$$

   Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ. Find θ.

Problems 3 and 4 is from J. J. Sakurai: *Modern Quantum Mechanics*, numbers 3.2 and 3.8, respectively.