QUANTUM MECHANICS II Spring 2012 **Exercise 3**

Session of Tuesday 7 February at 16-18 in aud A315

1. Consider the harmonic oscillator: $H = H_0 + V(x)$ with $H_0 = p^2/2m$ and $V(x) = \frac{1}{2}m\omega^2 x^2$. In the Interaction Picture (IP), the operator time dependence is defined by

$$x_{IP}(t) = e^{iH_0 t}\, x\, e^{-iH_0 t} \quad \text{and} \quad p_{IP}(t) = e^{iH_0 t}\, p\, e^{-iH_0 t}$$

whereas the states have the time dependence governed by

$$i\frac{d}{dt}|\psi, t\rangle = V_{IP}|\psi, t\rangle \quad \text{where} \quad V_{IP} = e^{iH_0 t}\, V\, e^{-iH_0 t}$$

   (a) Express the position and momentum operators $x_{IP}(t), p_{IP}(t)$ in terms of the (Schrödinger picture) operators x, p. You may use the general commutation relation you derived in problem 1 of Exercise 2.

   (b) Explain why the above definition of the Interaction Picture is not natural for the Harmonic Oscillator even for small ω, nor for any other potential $V(x)$ which confines the particle to a limited region.

2. (a) Express the Harmonic Oscillator V_{IP} (defined above) in terms of the (Schrödinger) operators x, p, using the general commutation relation you derived in problem 1 of Exercise 2.

   (b) Express V_{IP} in terms of $x_{IP}(t), p_{IP}(t)$. Could you have obtained this result in a simpler way?

3. The propagator for the harmonic oscillator potential is

$$K(x, t > 0; x_0, t = 0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}} \exp\left\{\frac{im\omega}{2\hbar\sin(\omega t)}\left[(x^2 + x_0^2)\cos(\omega t) - 2xx_0\right]\right\}$$

and $K(x, t < 0; x_0, 0) = 0$.

   (a) Show that $\lim_{t\to 0^+} K(x, t; x_0, 0) = \delta(x - x_0)$.
   Hint: Use $\lim_{\varepsilon\to 0} \exp(ix^2/\varepsilon)/\sqrt{i\pi\varepsilon} = \delta(x)$.

   (b) Show that for $t > 0$ K satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}K(x, t; x_0, 0) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}m\omega^2 x^2\right] K(x, t; x_0, 0)$$

4. (a) According to the definition of the propagator, the wave function at time t is obtained from the one at $t = 0$ as $\psi(x, t) = \int dx_0 K(x, t; x_0, 0)\psi(x_0, t = 0)$. Calculate $\psi(x, t)$ in the case of the harmonic oscillator, when $\psi(x_0, t = 0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x_0^2\right)$ is the ground state wave function. Motivate your result.

   (b) Calculate the propagator trace $G(t) \equiv \int_{-\infty}^{\infty} dx K(x, t; x, 0)$ for the harmonic oscillator and verify that $G(t) = \sum_n \exp(-iE_n t/\hbar)$.

1. a) $x_{IP}(t) = e^{i\frac{p^2}{2m}t} \, x \, e^{-i\frac{p^2}{2m}t}$

Exc. 2.1: $[x, e^{-i\frac{p^2}{2m}t}] = i\frac{\partial}{\partial p} e^{-i\frac{p^2}{2m}t} = +\frac{p}{m} t \, e^{-i\frac{p^2}{2m}t}$

$$\hat{x}_{IP}(t) = \hat{x} + \frac{\hat{p}}{m} t$$

$$[p, e^{-i\frac{p^2}{2m}t}] = 0 \Rightarrow p_{IP}(t) = \hat{p}$$

b) $\hat{x}_{IP}$ increases linearly with t. This is appropriate only for a particle which is not bound to a finite region of x.

2. $V_{IP} = \frac{1}{2} m\omega^2 e^{i\frac{\hat{p}^2}{2m}t} \hat{x}^2 e^{-i\frac{\hat{p}^2}{2m}t}$

a) $[x^2, e^{-i\frac{p^2}{2m}t}] = x[x, e^{-i\frac{p^2}{2m}t}] + [x, e^{-i\frac{p^2}{2m}t}]x$

$$= \frac{t}{m} xp \, e^{-i\frac{p^2}{2m}t} + \frac{t}{m} p \, e^{-i\frac{p^2}{2m}t} x$$

$$= \frac{t}{m} e^{-i\frac{p^2}{2m}t} \left(xp + \frac{p^2}{m}t + px\right)$$

$$V_{IP} = \frac{1}{2} m\omega^2 \left\{ x^2 + \frac{t}{m}(xp + px) + \frac{t^2}{m^2} p^2 \right\}$$

b) $V_{IP} = \frac{1}{2} m\omega^2 \left(x + \frac{t}{m} p\right)^2 = \frac{1}{2} m\omega^2 \hat{x}_{IP}(t)$

This result could have been obtained immediately:

$$e^{iH_0 t} \hat{x}^n e^{-iH_0 t} = (\hat{x}_{IP})^n$$

Hence through a Tailor expansion:

$$e^{iH_0 t} V(\hat{x}) e^{-iH_0 t} = V(\hat{x}_{IP})$$

3 a) $t \to 0$ limit of K

$$\lim_{t\to 0} K(x,t;0,0) = \sqrt{\frac{m}{2\pi i\hbar t}}\ \exp\left[i\frac{m}{2\hbar t}x^2\right] = \delta(x)$$

Since only $x \simeq 0$ contributes, and

$$\int_{-\infty}^{\infty} dx\ \exp\left(i\frac{mx^2}{2\hbar t}\right) = \sqrt{\frac{2\hbar t}{-im}}\sqrt{\pi} = \sqrt{\frac{2\pi i\hbar t}{m}}$$

Thus K is indep. of t at low $t>0$

It is ill-defined for $t=0$

If we define $K(t<0)=0 \Rightarrow \Theta(t)$.

b) $$i\hbar\frac{\partial K}{\partial t} = i\hbar K\left\{-\frac{\omega}{2}\frac{\cos\omega t}{\sin\omega t} - \frac{im\omega^2(x^2+x_0^2)}{2\hbar\sin^2\omega t}\left[\cos^2\omega t+\sin^2\omega t\right] + \frac{im\omega^2 xx_0}{\hbar\sin^2\omega t}\cos\omega t\right\}$$

$$= K\left\{-\frac{i\hbar\omega}{2}\frac{\cos\omega t}{\sin\omega t} + \frac{m\omega^2(x^2+x_0^2)}{2\sin^2\omega t}\cos^2\omega t + \frac{1}{2}m\omega^2(x^2+x_0^2) - \frac{m\omega^2 xx_0}{\sin^2\omega t}\cos\omega t\right\}$$

$$\frac{\partial}{\partial x}K = \frac{im\omega}{\hbar\sin\omega t}(x\cos\omega t - x_0)K$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}K = K\frac{m\omega^2}{2\sin^2\omega t}(x\cos\omega t-x_0)^2 - \frac{i\hbar\omega}{2\sin\omega t}\cos\omega t\, K$$

$$\Rightarrow\ i\hbar\frac{\partial K}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2\right]K$$

4a : Eigenfunction

$$\Psi(x,t) = \int dx_0\, K(x,t;x_0,0)\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}x_0^2\right]\cdot\left(\frac{m\omega}{\pi\hbar}\right)^{-1/4}$$

$$= \frac{m\omega}{\pi\hbar\sqrt{2i\sin\omega t}}\int dx_0 \exp\left\{\frac{im\omega\cos\omega t}{2\hbar\sin\omega t}x^2 - \frac{m\omega}{2\hbar}\left(1-\frac{i\cos\omega t}{\sin\omega t}\right) *\right.$$

$$\left. * \left[x_0 + \frac{ix}{\sin\omega t\left(1-\frac{i\cos\omega t}{\sin\omega t}\right)}\right]^2 - \frac{m\omega x^2}{2\hbar\sin^2\omega t}\frac{1}{\left(1-\frac{i\cos\omega t}{\sin\omega t}\right)}\right\}\left(\frac{m\omega}{\pi\hbar}\right)^{-1/4}$$

$$= \frac{m\omega}{\pi\hbar\sqrt{2i\sin\omega t}}\sqrt{\frac{2\hbar\sin\omega t}{m\omega(-i)e^{+i\omega t}}}\sqrt{\pi}\exp\left\{-x^2\frac{m\omega}{2\hbar}\frac{1}{\sin\omega t} *\right.$$

$$\left. * \left[\frac{1}{-ie^{i\omega t}} - i\cos\omega t = \sin\omega t\right]\right\} * \left(\frac{m\omega}{\pi\hbar}\right)^{-1/4}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left\{-x^2\frac{m\omega}{2\hbar}\right\}e^{-i\omega t/2} \text{ ok.}$$

4b Trace $\quad G(t) = \int dx\, K(x,t;x,0)$

$$V=0: \quad K(x,t;x,0) = \sqrt{\frac{m}{2\pi i\hbar t}}$$

$$G(t) = L\cdot\sqrt{\frac{m}{2\pi i\hbar t}} \quad \text{for} \quad \int_{-\infty}^{\infty} dx = L$$

$$\text{Cf.} \quad \int \frac{dp}{2\pi\hbar} e^{-ip^2 t/2m\hbar} = \qquad u = \sqrt{\frac{it}{2m\hbar}}\, p$$

$$= \frac{1}{2\pi\hbar}\sqrt{\frac{2m\hbar}{it}}\left\{\int_{-\infty}^{\infty} du\, e^{-u^2} = \int_0^{\infty}\frac{d(u^2)}{\sqrt{u^2}} e^{-u^2} = \sqrt{\pi}\right\} = \sqrt{\frac{m}{i2\pi\hbar t}}$$

$$V = \frac{1}{2}m\omega^2 x^2$$

$$G(t) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega t)}}\int_{-\infty}^{\infty} dx\, \exp\left[\frac{im\omega}{2\hbar\sin\omega t}2(\cos\omega t-1)x^2\right]$$

$$= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}\sqrt{\frac{2\hbar\sin\omega t}{-im\omega\, 2(\cos\omega t-1)}}\sqrt{\pi} = \frac{1}{\sqrt{2\underbrace{(\cos\omega t-1)}_{-2\sin^2\omega t/2}}} =$$

$$= \frac{1}{2i\sin(\omega t/2)}$$

$$\text{Cf.} \quad \sum_{n=0} e^{-i\hbar\omega t(n+1/2)/\hbar} = \frac{e^{-i\omega t/2}}{1-e^{-i\omega t}} = \frac{1}{2i\sin(\omega t/2)}$$

$$\tilde{G}(E) = -\frac{i}{\hbar}\int_0^{\infty} dt \sum_{n=0}^{\infty} e^{-i\overbrace{E_n}^{-i\varepsilon} t/\hbar}\, e^{iEt/\hbar} = -\frac{i}{\hbar}\sum_{n=0}^{\infty}\int_0^{\infty} dt\, e^{i(E-E_n+i\varepsilon)t/\hbar}$$

$$= \sum_{n=0}^{\infty}\frac{1}{E-E_n+i\varepsilon}$$