Degenerate Quantum Gases 2006

Lecture 14: Superfluid Fermion gases

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BCS theory and cold gases

- Two-component Fermi gas where components interact through s-wave interaction with strength g
- $g = 4\pi\hbar^2 a/m$, but note that in reality we should use the effective interaction which includes a correction to the above "contact" or δ-function interaction. This renormalizes away certain divergences later on.
- Hamiltonian we are interested in is thus

$$\begin{aligned} H &= \int d^3r \sum_{\sigma} \hat{\psi}^{\dagger}_{\sigma}(\mathbf{x},t) \left[-\frac{\hbar^2\nabla^2}{2m} - \mu_{\sigma} \right] \hat{\psi}_{\sigma}(\mathbf{x},t) \\ &+ g\hat{\psi}^{\dagger}_{\uparrow}(\mathbf{x},t)\hat{\psi}^{\dagger}_{\downarrow}(\mathbf{x},t)\hat{\psi}_{\downarrow}(\mathbf{x},t)\hat{\psi}_{\uparrow}(\mathbf{x},t) \end{aligned} \quad (1)$$

BCS theory and cold gases

- Field operators are now fermionic and that implies they have **anti**-commutation relations

$$\begin{aligned}\{\psi_\sigma(x,t),\psi^\dagger_{\sigma'}(x',t')\} &= \psi_\sigma(x,t)\psi^\dagger_{\sigma'}(x',t')+\psi^\dagger_{\sigma'}(x',t')\psi_\sigma(x,t) \\ &= \delta(x-x')\delta(t-t')\delta_{\sigma,\sigma'} \qquad (2)\end{aligned}$$

$$\{\psi_\sigma(x,t),\psi_{\sigma'}(x',t')\} = \{\psi^\dagger_\sigma(x,t),\psi^\dagger_{\sigma'}(x',t')\} = 0 \qquad (3)$$

- We will focus on systems with homogeneous density and then it is more natural to work in momentum space...

$$\{\psi_{\mathbf{k},\sigma},\psi^\dagger_{\mathbf{k}',\sigma'}\} = \delta_{k,k'}\delta_{\sigma,\sigma'} \qquad (4)$$

BCS theory and cold gases

- Anti-commutation relations imply, that one MUST be careful with the operator orderings. Swapping the order of operator introduces a minus sign.
- Bosons are more forgiving in this respect.
- For ideal gas of fermions at $T = 0$ we fill accessible states from $\epsilon_k = 0$ up to level corresponding to the Fermi-energy, which is the chemical potential at $T = 0$

$$\begin{aligned} N_\sigma &= \sum_k \theta(\epsilon_F - \epsilon_k) \rightarrow \frac{V}{8\pi^3} \int d^3k\theta(\epsilon_F - \epsilon_k) \\ &= \frac{V}{2\pi^2} \int dkk^2\theta(\epsilon_F - \epsilon_k) \end{aligned} \tag{5}$$

BCS theory and cold gases

- Changing the variable to $\epsilon = \hbar^2 k^2/2m$

$$N_\sigma = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^{\epsilon_F} d\epsilon\sqrt{\epsilon} = \frac{V}{3\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2} \quad (6)$$

- Therefore the Fermi-energy is associated with the density through

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2(6\pi^2 n_\sigma)^{2/3}}{2m} \quad (7)$$

- Quantum degeneracy sets in when the temperature becomes $k_B T \sim \epsilon_F$... at higher temperatures classical approximation is a reasonable approximation.

Cooper problem

- Think of two-fermions interacting in the presence of the Fermi-sea...Spin part of the wavefunction is a singlet and consider the spatial part of the two-body wavefunction

$$\psi(r_1 - r_2) = \sum_{k>k_F} g_k \cos(\mathbf{k}\cdot(r_1 - r_2)) \tag{8}$$

- Sum is restricted to above the Fermi surface since atoms cannot occupy levels in the Fermi sea.
- Insert this into the Schröringer equation

$$(E - 2\epsilon_k)g_k = \sum_{k'>k_F} V_{k,k'} g_{k'} \tag{9}$$

where $V_{k,k'} \sim \int V(\mathbf{r}) e^{i(k-k')\mathbf{r}} d\mathbf{r}$

Cooper problem

- Are there states with energy $E < 2E_F$? If so, bound-pair state can exist
- Take $V_{k,k'} = -U$ for states out to a cutoff energy $\hbar\omega_c$ away from E_F and zero otherwise

$$g_k = U \frac{\sum g_{k'}}{2\epsilon_k - E} \tag{10}$$

- Sum both sides and cancel terms...

$$\frac{1}{U} = \sum_{k>k_F} (2\epsilon_k - E)^{-1} \tag{11}$$

Cooper problem

- Replace sum with an integral, $N(0)$ is density of states at the Fermi level

$$\begin{aligned} \frac{1}{U} &= N(0) \int_{E_F}^{E_F+\hbar\omega_c} \frac{d\epsilon}{2\epsilon - E} \\ &= \frac{N(0)}{2} \ln \frac{2E_F - E + 2\hbar\omega_c}{2E_F - E} \end{aligned} \qquad (12)$$

- with weak-coupling $N(0)U \ll 1$ and

$$E = 2E_F - 2\hbar\omega_c e^{-2/N(0)U} \qquad (13)$$

- Bound-pair state exists for arbitrarily small couplings... The Fermi sea is then unstable towards pair formation.

Mean-field theory

- Fermions cannot Bose condense due to their statistics. However, pairs of Fermions can behave somewhat like bosons, especially if the pair is small compared to the separations between atoms
- Lets construct a mean-field theory using

$$\psi_{-k,\downarrow}\psi_{k,\uparrow} = b_k + \left(\psi_{-k,\downarrow}\psi_{k,\uparrow} - b_k\right) \tag{14}$$

where $b_k = \langle \psi_{-k,\downarrow}\psi_{k,\uparrow} \rangle$ and by assuming that the last fluctuation term is small.

- Note that we have assumed here that the pair has zero momentum i.e. the interaction term is approximated as

$$H_{I,BCS} = \frac{g}{V} \sum_{k,k'} \psi^{\dagger}_{k,\uparrow}\psi^{\dagger}_{-k,\downarrow}\psi_{-k',\downarrow}\psi_{k',\uparrow} \tag{15}$$

Mean-field theory

- Generally other terms do exist since, when you Fourier transform a product of 4 operators into momentum space you get 4 k-indices and after spatial integration you get 1 δ-function which still leaves 3-independent k-indices...not just two as in this approximation.
- Using this mean-field approach we find the **quadratic** model Hamiltonian

$$
\begin{aligned}
H &= \sum_{k,\sigma} (\epsilon_k - \mu_\sigma) \psi^\dagger_{k,\sigma} \psi_{k,\sigma} \\
&+ \frac{g}{V} \sum_{k,k'} \psi^\dagger_{k,\uparrow} \psi^\dagger_{-k,\downarrow} b_{k'} + b^*_k \psi_{-k',\downarrow} \psi_{k',\uparrow} - b^*_k b_{k'}
\end{aligned}
\tag{16}
$$

- The task is now to find b_k:s self-consistently

Mean-field theory

- For later convenience I define a new quantity

$$\Delta = \frac{g}{V}\sum_k \langle \psi_{-k,\downarrow}\psi_{k,\uparrow}\rangle \tag{17}$$

which has units of energy... This is nothing but

$$\Delta = g\langle \hat{\psi}_\downarrow(\mathbf{x},t)\hat{\psi}_\uparrow(\mathbf{x},t)\rangle \tag{18}$$

in the original field operators and the assumption about zero momentum pairs therefore means assumption that $\langle \hat{\psi}_\downarrow(\mathbf{x},t)\hat{\psi}_\uparrow(\mathbf{x},t)\rangle = constant$

Mean-field theory

- The mean-field Hamiltonian is now:

$$
\begin{aligned}
H &= \sum_{k,\sigma} (\epsilon_k - \mu_\sigma) \psi^\dagger_{k,\sigma} \psi_{k,\sigma} \\
&+ \Delta \psi^\dagger_{k,\uparrow} \psi^\dagger_{-k,\downarrow} + \Delta^* \psi_{k,\downarrow} \psi_{-k,\uparrow} - \frac{V\Delta^2}{g}
\end{aligned}
\tag{19}
$$

- This we can write as

$$
H = -\frac{V\Delta^2}{g} + \sum_k \xi_{k,\uparrow} + \begin{bmatrix} \psi^\dagger_{k,\downarrow} \\ \psi_{-k,\uparrow} \end{bmatrix} \begin{bmatrix} \xi_{k,\downarrow} & \Delta \\ \Delta^* & -\xi_{k,\uparrow} \end{bmatrix} \begin{bmatrix} \psi_{k,\downarrow} \\ \psi^\dagger_{-k,\uparrow} \end{bmatrix}
$$

where $\xi_\sigma = \epsilon_k - \mu_\sigma$

Mean-field theory

- We can diagonalize this problem and thus get a canonical transformation into a basis in which the Hamiltonian has a structure of an ideal gas Hamiltonian

$$\psi_{k,\downarrow} = u_k \gamma_{k,\downarrow} + v_k^* \gamma^\dagger_{-k,\uparrow} \tag{20}$$

$$\psi^\dagger_{-k,\uparrow} = -v_k \gamma_{k,\downarrow} + u_k \gamma^\dagger_{-k,\uparrow} \tag{21}$$

where $|u_k|^2 + |v_k|^2 = 1$ when transformation is canonical (preserves anti-commutation relations)

- In the case that densities are equal $\mu_\downarrow = \mu_\uparrow = \mu$ and we find dispersions

$$E_k = \pm E_{BCS} = \pm\sqrt{\xi_k^2 + \Delta^2} \tag{22}$$

Mean-field theory

- While the Bogoliubov amplitudes are

$$|v_k|^2 = 1 - |u_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right) \qquad (23)$$

- For an ideal gas $\Delta = 0$ and then $v_k = 0$ if $\epsilon_k > \mu$ and $v_k = 1$ if $\epsilon_k < \mu$.
- On the other hand the density is

$$\begin{aligned} \langle n_{k,\downarrow}\rangle &= \langle \psi^\dagger_{k,\downarrow}\psi_{k,\downarrow}\rangle = u_k^2\langle\gamma^\dagger_{k,\downarrow}\gamma_{k,\downarrow}\rangle + v_k^2\langle\gamma_{-k,\uparrow}\gamma^\dagger_{-k,\uparrow}\rangle \\ &= u_k^2 n_F(E_{BCS}) + v_k^2(1 - n_F(E_{BCS})) = v_k^2 \end{aligned}$$

where on the last step we took the limit $T \to 0$

- Sum over the k:s and get the total atom number...Agrees with the ideal gas result derived earlier!

Mean-field theory

- The dispersion E_{BCS} has a minimum value of Δ, i.e. there is a gap. (Δ is known as the gap parameter/order parameter)
- How do we calculate the value of Δ? Insert the transformation into the definition...

$$\Delta = \frac{g}{V} \sum_k \langle \psi_{-k,\downarrow} \psi_{k,\uparrow} \rangle \tag{24}$$

- In the diagonal basis $\langle \gamma\gamma \rangle = 0 = \langle \gamma^\dagger \gamma^\dagger \rangle$ so we find

$$\Delta = \frac{g}{V} \sum_k -u_k v_k^* (1 - 2n_F(E_{BCS})) = \frac{-g\Delta}{V} \sum_k \frac{1 - 2n_F(E_{BCS})}{2E_{BCS}}$$

Mean-field theory

- So we have an integral equation for Δ

$$\frac{-1}{g} = \frac{1}{V}\sum_k \frac{1-2n_F(E_{BCS})}{2E_{BCS}} = \frac{1}{(2\pi)^3}\int d^3k \frac{1-2n_F(E_{BCS})}{2\sqrt{\xi_k+\Delta^2}}$$

- This ALWAYS has a solution at $T=0$ if $g<0$
- The integral is ultraviolet divergent (at large energies the integrand is $\sim \sqrt{\epsilon}/\epsilon = 1/\sqrt{\epsilon}$)
- This divergence is caused by the contact interaction and using the proper effective interaction the coupling must be renormalized

$$\frac{-1}{g} = \frac{1}{(2\pi)^3}\int d^3k \frac{1-2n_F(E_{BCS})}{2\sqrt{\xi_k+\Delta^2}} - \frac{1}{2\epsilon_k} \qquad (25)$$

converges and is cut-off independent.

Mean-field theory: number equation

- We still need the equation of state i.e. connect the particle density with the chemical potential
- We can use the result earlier...

$$
\begin{aligned}
n = \frac{N_\downarrow}{V} &= \frac{1}{V}\sum_k \langle n_{k,\downarrow}\rangle = \frac{1}{V}\sum_k \langle \psi^\dagger_{k,\downarrow}\psi_{k,\downarrow}\rangle \\
&= \frac{1}{V}\sum_k u_k^2 n_F(E_{BCS}) + v_k^2(1 - n_F(E_{BCS}))
\end{aligned}
$$

- One can show that BCS-solution has a lower energy than the ideal gas when non-zero solution of the gap-equation can be found.

Size of the Cooper pairs

- Operator $\psi_{-k,\downarrow}\psi_{k,\uparrow}$ annihilates a Cooper-pair
- The Δ gives the width of the region where v_k drops to zero in the vicinity of μ. I.e. it defines the characteristic range of k values involved in Cooper pairing
- Using only natural constants, Δ and ϵ_F we can form an associated length scale

$$\xi = \sqrt{2\epsilon_F \hbar^2/m}\frac{1}{\Delta} \tag{26}$$

- In the weak coupling regime we can show that is simply related to the size of the Cooper pairs... Small Δ implies large pair size

Mean-field theory: BCS limit

- BCS limit of weak interactions means $1/k_F a \to -\infty$ and then at $T = 0$

$$\Delta(T=0) \sim \epsilon_F \exp(-\frac{\pi}{2k_F|a|}) \quad (27)$$

- Note that this expression is not analytic at $a = 0$...The result could not have been achieved using perturbation theory on an ideal Fermi gas.
- As the temperature increases there is a critical temperature where gap vanishes

$$T_c \simeq \frac{8\epsilon_F}{k_B\pi} e^{\gamma-2} \exp\left(\frac{-\pi}{2k_F|a|}\right) \sim \Delta(T=0)/k_B \quad (28)$$

Mean-field theory: BCS limit

- What about chemical potential in the BCS limit?
- Solving the number equation it turns out that the chemical potential is not changed by much from its ideal gas value when $1/|k_F a| \gg 1$
- The inter-atomic distance $l \sim 1/n^{1/3}$...This implies that the Cooper-pair size

$$\frac{\xi}{l} \sim \frac{\epsilon_F}{\Delta} \gg 1 \qquad (29)$$

is much larger than the separation between atoms

Mean-field theory: BEC limit

- In the BEC/strong coupling limit $1/k_F a \to +\infty$... You can still solve the gap and the number equations
- Chemical potential is strongly modified from the ideal gas value

$$\mu_{BEC} = -\frac{E_b}{2} + 2\epsilon_F(k_F a)/3\pi \qquad (30)$$

where $E_b = \hbar^2/ma^2$ is the pair binding energy.

- Note: when $a > 0$ a two-body bound-state exists in the scattering problem!
- The order parameter becomes

$$\Delta_{BEC} = \sqrt{16/\pi}\epsilon_F/\sqrt{k_F a} \qquad (31)$$

BCS many-body wavefunction

- The mean-field theory outlined here turns out to correspond to a wavefunction ansatz of form

$$|\Psi_{BCS}\rangle = \prod_{\mathbf{k}} \left(u_k + v_k \psi^\dagger_{k,\uparrow} \psi^\dagger_{-k,\downarrow} \right) |0\rangle \tag{32}$$

- Here the probability that the pair $(k\uparrow, -k\downarrow)$ is occupied is $|v_k|^2$ while its probability to be unoccupied is $1 - |v_k|^2 = |u_k|^2$
- This was the approach used by the original Bardeen-Cooper-Schrieffer trio.
- Insert this ansatz into the Hamiltonian and find the optimal values for u_k and v_k (and fix the average density)

BCS-BEC cross-over

- In the BEC limit the pair size becomes smaller than the separation between atoms.
- Cooper pairs become "real" molecules which can then Bose condense....By changing a from the BCS to BEC regime we realize the **BCS-BEC cross-over**
- Discussion here is **valid for wide Feshbach resonance**, for which the likelihood of being on the molecular state associated with the closed channel of the scattering problem is very small.
- For wide resonance the single-channel approach outlined here is sensible (This is usually the case)
- **For narrow resonances, the closed channel molecule cannot be ignored** and we should incorporate it into our theory explicitly.

BCS-BEC cross-over

- In the BCS regime the critical temperature depends exponentially on the coupling strength... It becomes extremely low and cannot be reached in practice
- However, on resonance the solution of the gap equation and number equations reveal that the critical temperature there can be a substantial fraction of the Fermi-energy ($E_F \sim 1\,\mu K$)
- In the strong-coupling region cooling can easily take us into the pairing regime
- Note: In these systems $T_c/T_F \sim 1$, so in this sense these are truly high temperature superfluids. High T_c superconductors have $T_c/T_F \sim 10^{-3} \ldots 10^{-2}$

Fermion Superfluidity: Experiments

Few months after having reported a molecular condensate of ^{40}K atoms D. Jin et al. observed hints (2004) about non-zero condensate fraction also at the BCS side of the resonance. (Sweep experiment complicated the understanding of physics.)

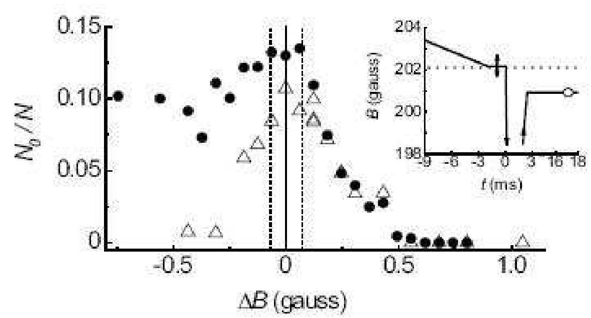

FIG. 2: Measured condensate fraction as a function of detuning from the Feshbach resonance $\Delta B = B_{hold} - B_0$. Data here were taken for $t_{hold} = 2$ ms (●) and $t_{hold} = 30$ ms (△) with an initial cloud at $T/T_F = 0.08$ and T_F=0.35 μK. The area between the dashed lines around $\Delta B = 0$ reflects the uncertainty in the Feshbach resonance position based on the 10% − 90% width of the feature in Fig. 1. Condensation of fermionic atom pairs is seen near and on either side of the Feshbach resonance. Comparison of the data taken with the different hold times indicates that the pair condensed state has a significantly longer lifetime near the Feshbach resonance and on the BCS ($\Delta B > 0$) side. The inset shows a schematic of a typical magnetic-field sweep used to measure the fermionic condensate fraction. The system is first prepared by a slow magnetic-field sweep towards the resonance (dotted line) to a variable position B_{hold}, indicated by the two-sided arrow. After a time t_{hold} the optical trap is turned off and the magnetic field is quickly lowered by ∼ 10 G to project the atom

Fermion Superfluidity: Experiments

Collective excitations were studied across the cross-over region and results compared with the superfluid hydrodynamics theory... Grimm et al.(Innsbruck, 2004)

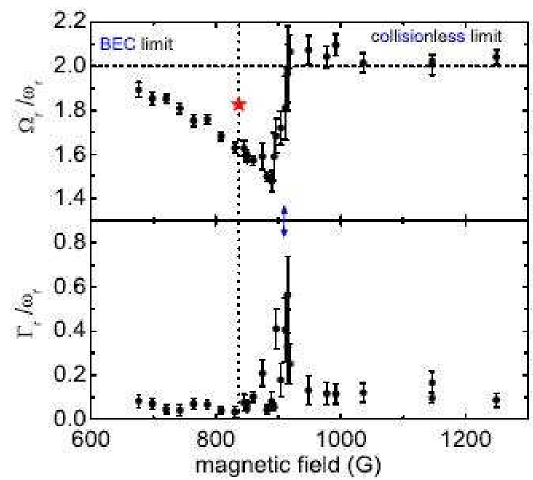

FIG. 3: Measured frequency Ω_r and damping rate Γ_r of the radial compression mode, normalized to the trap frequency (sloshing mode frequency) ω_r. In the upper graph, the dashed line indicates $\Omega_r/\omega_r = 2$, which corresponds to both the BEC limit and the collisionless Fermi gas limit. The vertical dotted line marks the resonance position at 837(5) G. The star indicates the theoretical expectation of $\Omega_r/\omega_r = \sqrt{10/3}$ in the unitarity limit. A striking change in the excitation frequency occurs at ~910 G (arrow) and is accompanied by anomalously strong damping.

Fermion Superfluidity: Experiments

Collective excitations Thomas et al.(Duke University, 2004)

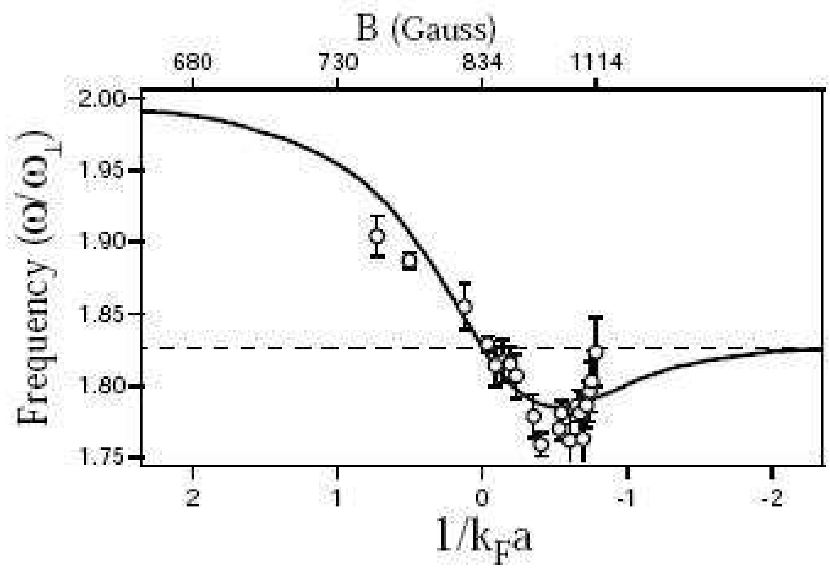

FIG. 1: Magnetic field dependence of the frequency ω of the radial breathing mode. Solid line is the theory based on superfluid hydrodynamics from Hu et al. [17]. Dashed line is the hydrodynamic frequency for a unitarity gas, $\sqrt{10/3}$, as predicted at resonance. Note that the top (magnetic field) axis is not linear.

Fermion Superfluidity: Experiments

Innsbruck group then proceeded to measure the single particle excitation gap using RF-spectroscopy. Direct observation of the pairing gap, Science 2004 !

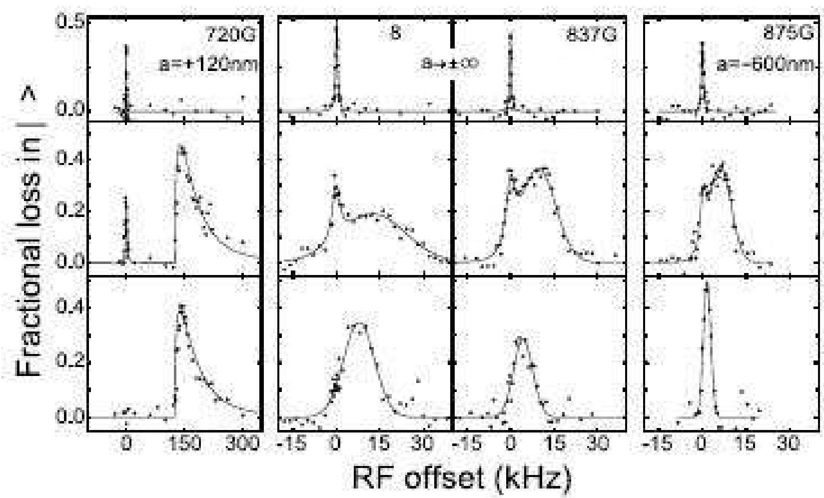

Fig. 1. RF spectra for various magnetic fields and different degrees of evaporative cooling. The RF offset ($k_B \times 1\,\mu K \simeq h \times 20.8$ kHz) is given relative to the atomic transition $|2\rangle \rightarrow |3\rangle$. The molecular limit is realized for $B = 720$ G (first column). The resonance regime is studied for $B = 822$ G and 837 G (second and third column). The data at 875 G (fourth column) explore the crossover on the BCS side. Upper row, signals of unpaired atoms at $T' \approx 6T_F$ ($T_F = 15\,\mu K$); middle row, signals for a mixture of unpaired and paired atoms at $T' = 0.5T_F$ ($T_F = 3.4\,\mu K$); lower row, signals for paired atoms at $T' < 0.2T_F$ ($T_F = 1.2\,\mu K$). Note that the true temperature T of the atomic Fermi gas is below the temperature T' which we measure in the BEC limit (see text). The solid lines are introduced to guide the eye.

Fermion Superfluidity: Experiments

Pairing gap itself could have phase fluctuations so strictly speaking superfluidity was not yet established by the Grimm experiment. Ketterle et al. (MIT, 2005) observed the vortices and thus the superfluidity was also established.

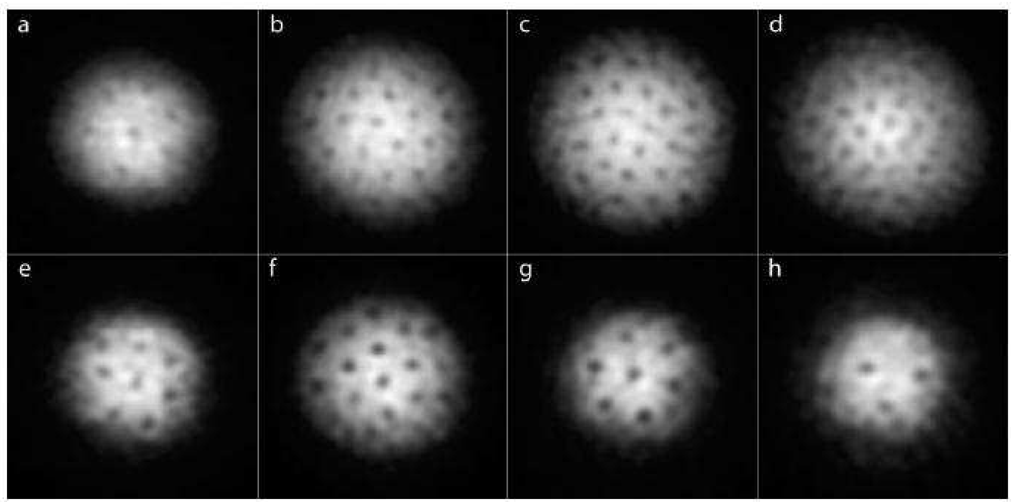

Fig. 2: Vortices in a strongly interacting gas of fermionic atoms on the BEC- and the BCS-side of the Feshbach resonance. At the given field, the cloud of lithium atoms was stirred for 300 ms (a) to 500 ms (b-h) followed by an equilibration time of 500 ms. After 2 ms of ballistic expansion, the magnetic field was ramped to 735 G for imaging (see text for details). The magnetic fields were (a) 740 G, (b) 766 G, (c) 792 G, (d) 812 G, (e) 833 G, (f) 843 G, (g) 853 G and (h) 863 G. The field of view of each image is $880\ \mu m \times 880\ \mu m$.

Fermion Superfluidity: Experiments

Ketterle et al.

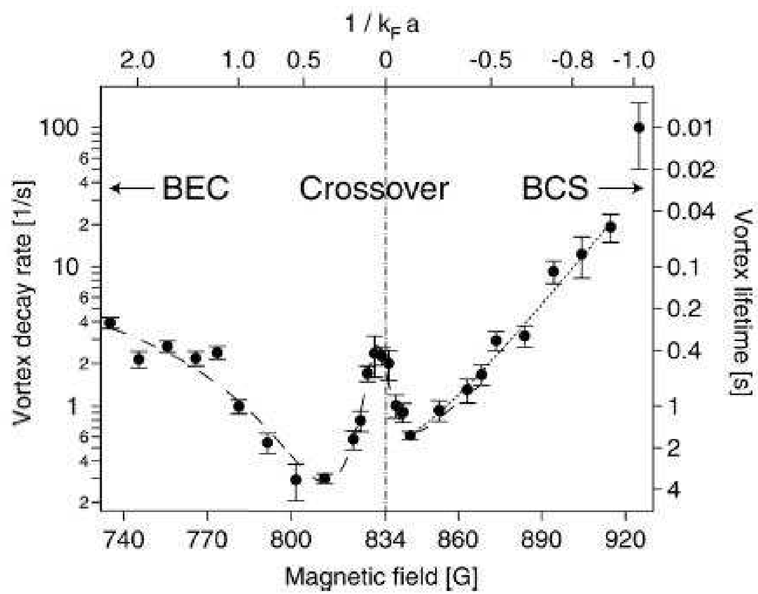

Fig. 7: Decay rate and lifetime of the vortex lattice vs magnetic field and interaction strength. The position of the Feshbach resonance[26] is marked with the dash-dotted line. The dashed and dotted lines are Gaussian and exponential fits to guide the eye, the dashed line including a lorentzian fit for the narrow feature near resonance.

Essays

- The deadline is **monday 4.9.2006**, but can return earlier...
- Essay can be written in english or finnish.
- How extensively the writer has read the relevant literature or papers? Essays based on a single source are worse than essay based on several sources.
- Extra points for making connection to other branches of physics.
- How well the author seems to understand the topic? Can he/she indicate new questions/problems?

Evaluation criteria continued...

- Presentation. The essay should be written with real sentences and not just a collection of formulas or an itemized list. Citations should naturally be used to indicate sources.
- 10 page essay is probably better than a 1 page essay, but 100 page essay is not necessarily better than the 10 page essay. Use common sense...
- Originality, initiative etc...:If you wish to solve some problems or examples on your own, by writing and using your own code, for example, I will take this into account in the overall evaluation. However, remember that such contributions should be clearly documented.

Possible topics for essays

1. Four wave mixing and non-linear effects in Bose-Einstein condensates
2. Quantum Hall physics in degenerate gases
3. Solitons in Bose-Einstein condensates
4. Feshbach resonance and tunability of interactions
5. Optical lattices
6. Non-classical states in a Bose gas
7. Vortex lattices in degenerate gases (Tomi Paananen)
8. Atom lasers
9. Disorder and ultra cold quantum gases
10. Others...feel free to suggest...