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# SCATTERING LENGTHS AND RESONANCE PARAMETERS FROM LATTICE SIMULATIONS

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- So far, there have been few studies of hadronic scattering on the lattice
- Scattering calculation requires 4-point functions
- Matrix elements and hadronic coupling constants have been studied in detail (3-point functions!)
- Scattering lengths:  $s$ -wave  $\pi$ - $\pi$  ,  $\pi$ - $N$  ,  $N$ - $N$  -scattering lengths have been calculated in quenched approximation
- No calculations with ‘full’ QCD
- Phase shift function  $\delta(p)$  and especially resonance parameters have been studied only using much simpler models — ‘toy’ models
- Very good results with some of the models — displaying the potential of lattice calculations

In this talk:

1. Short review of results from scattering length calculations using lattice QCD
2. How to calculate phase shift function and resonance parameters from the lattice [K.Rummukainen, S.Gottlieb, *Resonance scattering phase shifts on a non-rest-frame lattice*, Nucl.Phys. B450 (1995) 397]

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Problems with scattering calculations on the lattice (both QCD and the toy models):

- Typical lattice sizes  $\sim$  few times the interaction length! Impossible to observe freely propagating asymptotic states. In QCD,  $L \sim 1\text{--}3$  fm.
- The connection **real time scattering amplitude**  $\leftrightarrow$  **Euclidean lattice Green's functions** is very non-trivial. An elegant and 'exact' solution exists! The Lüscher formulas relate the phase shift to 2-particle energy levels measured from the Euclidean lattice.
- Four-point function calculation is quite troublesome, especially with QCD! Highly technical solutions.
- Problems with quenched approximation – pathological! [C.Bernard, M.Goltermann, hep-lat/9507004]

## Papers:

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### Scattering length with QCD (all quenched!):

- M.Guagnelli, E.Marinari, G.Parisi, Phys.Lett.240B, 188 (1990)  
 $\pi$ - $\pi$  and  $\pi$ - $N$  scattering lengths with Kogut-Susskind and Wilson fermion formulation, incomplete
  - S.R.Sharpe, R.Gupta, G.W.Kilcup, Nucl. Phys. B383, 309 (1992)  
 $\pi$ - $\pi$   $I = 2$  channel, Kogut-Susskind
  - R.Gupta, A.Patel, S.R.Sharpe, Phys. Rev. D48, 388 (1993)  
 $\pi$ - $\pi$   $I = 2$  channel, Wilson
  - M.Fukugita, Y.Kuramashi, M.Okawa, H.Mino, A.Ukawa, Phys. Rev. Lett. 71 (1993) 2387;  
Phys. Rev. Lett. 73 (1994) 2176;  
Phys. Rev. D52 (1995) 3003  
 $\pi$ - $\pi$   $I = 0$  and  $I = 2$ ,  $\pi$ - $N$ ,  $N$ - $N$  scattering lengths, Kogut-Susskind and Wilson formalisms
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Phase shift calculations (none with QCD):

- M.Lüscher, U.Wolff, Nucl. Phys. B339 (1990) 222  
1+1 -dimensional non-linear  $\sigma$  model
  - C.R.Gattringer and C.B.Lang, Nucl. Phys. B391 (1993) 463  
1+1 -dimensional; 2 coupled Ising spins
  - H.R.Fiebig, R.M.Woloshyn, A.Dominiguez, Nucl. Phys. B418 (1994) 649; H.R.Fiebig, O.Linsuain, H.Markum, K.Rabitsch, hep-lat/9508021  
2+1 -dimensional QED
  - M.Göckeler, H.A.Kastrup, J.Westphalen, F.Zimmermann, Nucl. Phys. B425 (1994) 413  
3+1 -dim.  $O(4)$ -symmetric  $\phi^4$
  - K.Rummukainen and S.Gottlieb, Nucl. Phys. B450 (1995) 397  
3+1 -dim. model, non-rest frame scattering
  - M.Göckeler, H.A.Kastrup, J.Viola, J.Westphalen, LATTICE 95, hep-lat/9509059  
1+1 -dim. Gross-Neveu model, fermion scattering
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- Most hadrons are resonances: for example,  $\pi\pi \leftrightarrow \rho$  in the  $L = 1, I = 1$  scattering channel ( $m_\rho = 770$  MeV,  $\Gamma_\rho = 151$  MeV).
- Resonances need to be taken into account in order to fully resolve the hadronic spectrum!
- Partial wave decomposition of the scattering amplitude:

$$T = \frac{16\pi W}{2ip} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) (e^{2i\delta_l} - 1)$$

where  $\delta_l$  is the phase shift, and  $W$  is the total energy of the scattering particles.

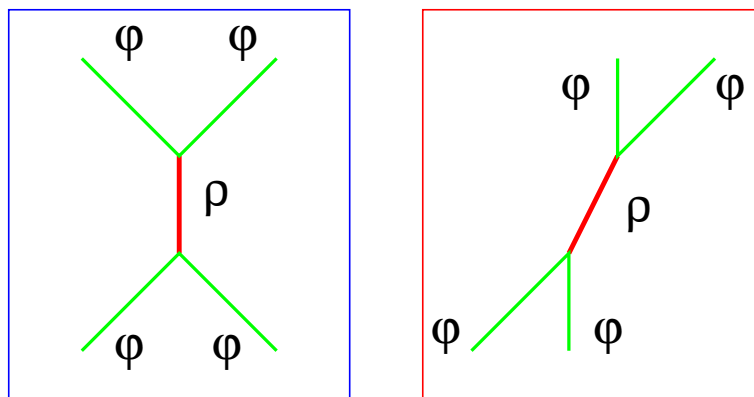
- **Problem:** lattice size  $\sim$  a few times the interaction length. Impossible to identify free asymptotic states.
- This has been addressed in a proposal by M. Lüscher [Commun.Math.Phys. 105 (1986); Nucl.Phys. B354 (1991)], giving a relation between the phase shift and the 2-particle energy spectrum measured from finite periodic lattices. The method requires that the scattering particles have opposite lattice momenta (zero total momentum  $\vec{P} = 0$ ).

We have generalized Lüscher's method to *arbitrary total momentum*  $\vec{P}$  [K.R., S.G, Nucl.Phys.B 450 (1995) 397]

- With  $\vec{P} \neq 0$ , the resonance can be observed using smaller volumes than with  $\vec{P} = 0$ .
- $\vec{P} = 0$  and  $\vec{P} \neq 0$  sectors can be measured from the same configurations.
- For a given lattice size, the two sectors yield the phase shift  $\delta_L(p)$  with different argument  $p$  – two data points from a single run!

To illustrate the difference in the required lattice size, consider a non-interacting system containing a light particle  $\phi$ , and a heavy “would-be resonance”  $\rho$ :

- $L$  = lattice size, lattice momenta  $k = n 2\pi/L$
- At the “would-be” resonance the energies are equal:  $W_\rho = W_{2\phi}$



- $\vec{P} = 0$ : the  $\phi$ 's have opposite momenta, at minimum  $p_\phi = (2\pi/L)$ , and  $p_\rho = 0$ :

$$\begin{aligned} W_\rho &= m_\rho \\ W_{2\phi} &= 2\sqrt{m_\phi^2 + p_\phi^2} \end{aligned}$$

Using  $W_\rho = W_{2\phi}$  and assuming, for simplicity,  $m_\rho = \frac{7}{3}m_\phi$ , this gives  $L \approx 10.5/m_\phi$ .

- $\vec{P} \neq 0$ :  $p_{\phi,1} = (2\pi/L)$ ,  $p_{\phi,2} = 0$ , and  $p_\rho = (2\pi/L)$ :

$$\begin{aligned} W_\rho &= \sqrt{m_\rho^2 + p_\rho^2} \\ W_{2\phi} &= m_\pi + \sqrt{m_\phi^2 + p_{\phi,1}^2} \end{aligned}$$

which yields  $L \approx 4.5/m_\phi$

- The ratio of volumes  $\frac{V_1}{V_2} = \left(\frac{10.5}{4.5}\right)^3 \approx 12.7!$  The non-zero momentum sector gains more than an order of magnitude over the zero momentum case.

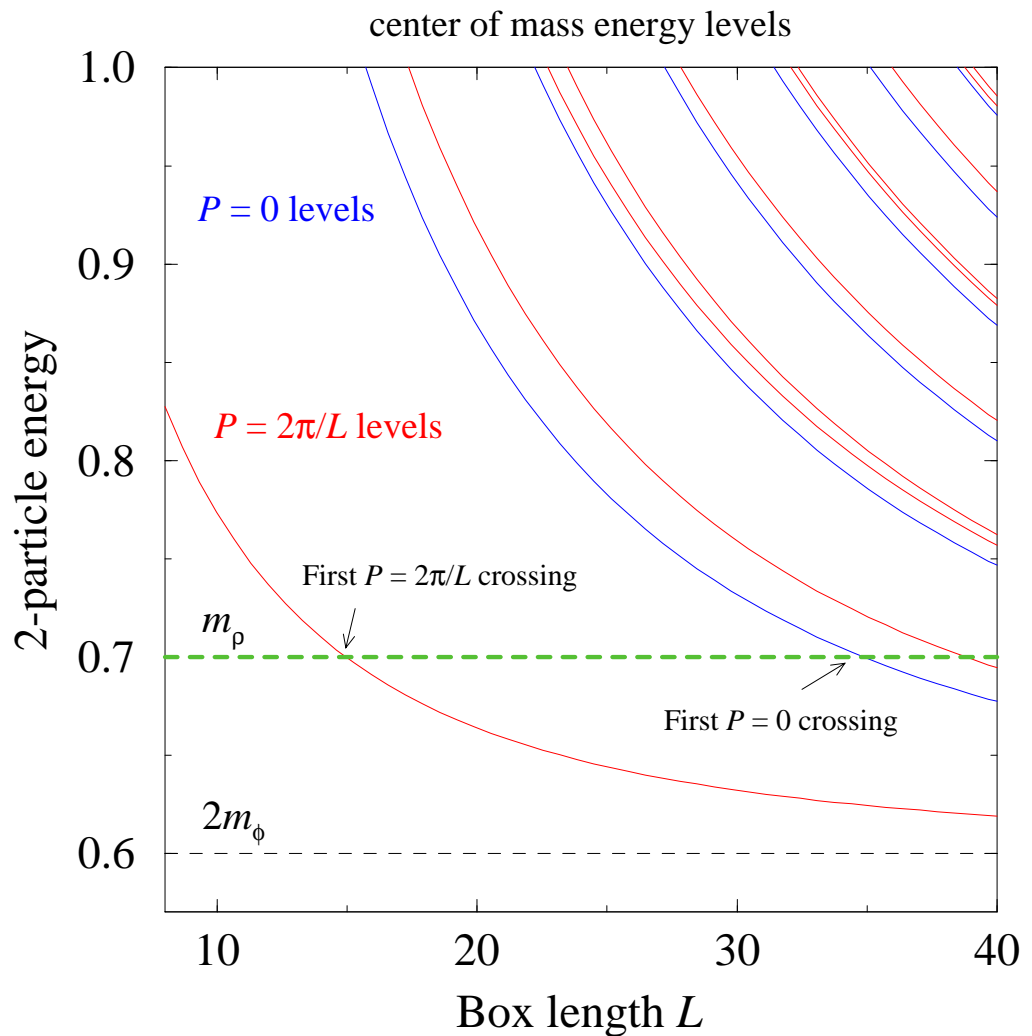
Even in  $\vec{P} \neq 0$  scattering, it is convenient to use the center of mass frame of the scattering particles. In this frame the 2-particle energy is, in terms of the lattice (laboratory) frame energy  $W_L$ ,

$$W_{\text{cm},2\phi} = \sqrt{W_{L,2\phi}^2 - \vec{P}^2}$$

and the momenta of the particles

$$p_{\text{cm},1} = p_{\text{cm},2} = \sqrt{\frac{1}{2}W_{\text{cm},2\phi} - m_\phi}$$

In the picture below, we show the 2-particle center of mass frame energy spectrum  $W_{2\phi}$  both for  $\vec{P} = 0$  (blue) and  $\vec{P} = 2\pi/L$  (red) sectors. The lattice sizes calculated in the previous example are indicated by arrows.



## Phase shift and energy spectrum

Let us consider *interacting* particles, and restrict in the case  $\vec{P} = 2\pi\mathbf{e}_1/L$ . The energy  $W_{\text{cm},2\phi}$  is measured, and the CM-frame momentum  $p_{\text{cm}}$  is obtained from

$$p_{\text{cm}} = \sqrt{W_{\text{cm},2\phi}^2 - m_\phi^2}. \quad (1)$$

In the s-wave ( $l = 0$ ) sector, the relation between  $p_{\text{cm}}$  and the phase shift  $\delta_0$  is given by

$$\delta_0(p_{\text{cm}}) = -\Phi(q) \bmod \pi, \quad q = \frac{p_{\text{cm}}L}{2\pi}, \quad (2)$$

where  $\Phi$  is a continuous function defined by

$$\boxed{\tan(-\Phi(q)) = \frac{\gamma q \pi^{3/2}}{Z_{00}(1; q^2)} \quad \Phi(0) = 0,} \quad (3)$$

where  $\gamma = (1 - v^2)^{-1/2} = (1 + \vec{P}^2/W_{\text{cm}}^2)^{1/2}$ . Function  $Z_{00}$  is a generalized zeta function, given by

$$Z_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{r} \in A} (r^2 - q^2)^{-s}, \quad (4)$$

where the set  $A$  is

$$A = \{\vec{r} \in \mathbf{R}^3 | r_1 = \gamma^{-1}(n_1 + 1/2), r_2 = n_2, r_3 = n_3; \vec{n} \in \mathbf{Z}^3\}.$$

The expansion of  $Z_{00}$  is convergent when  $\text{Re } s > 3/2$ , but it can be analytically continued to  $s = 1$ .

When  $\vec{P} \rightarrow 0$ ,  $\gamma \rightarrow 1$  and  $A \rightarrow \mathbf{Z}^3$ , and Lüscher's formula is recovered.

## Monte Carlo simulations

The method is tested with a simple 3+1 dim. spin model:<sup>1</sup>

$$S = -\kappa_\phi \sum_{x; \hat{\mu}} \phi_x \phi_{x+\hat{\mu}} - \kappa_\rho \sum_{x; \hat{\mu}} \rho_x \rho_{x+\hat{\mu}} + g \sum_{x; \hat{\mu}} \rho_x \phi_x \phi_{x+\hat{\mu}}.$$

Field variables  $\phi$  and  $\rho$  are restricted to  $\{-1, 1\}$ . Choosing the coupling constants suitably  $\rho$  appears as a resonance in s-wave  $\phi\phi$  scattering. We used 3 sets of couplings (A, B, C), given in the table below, together with the final results:

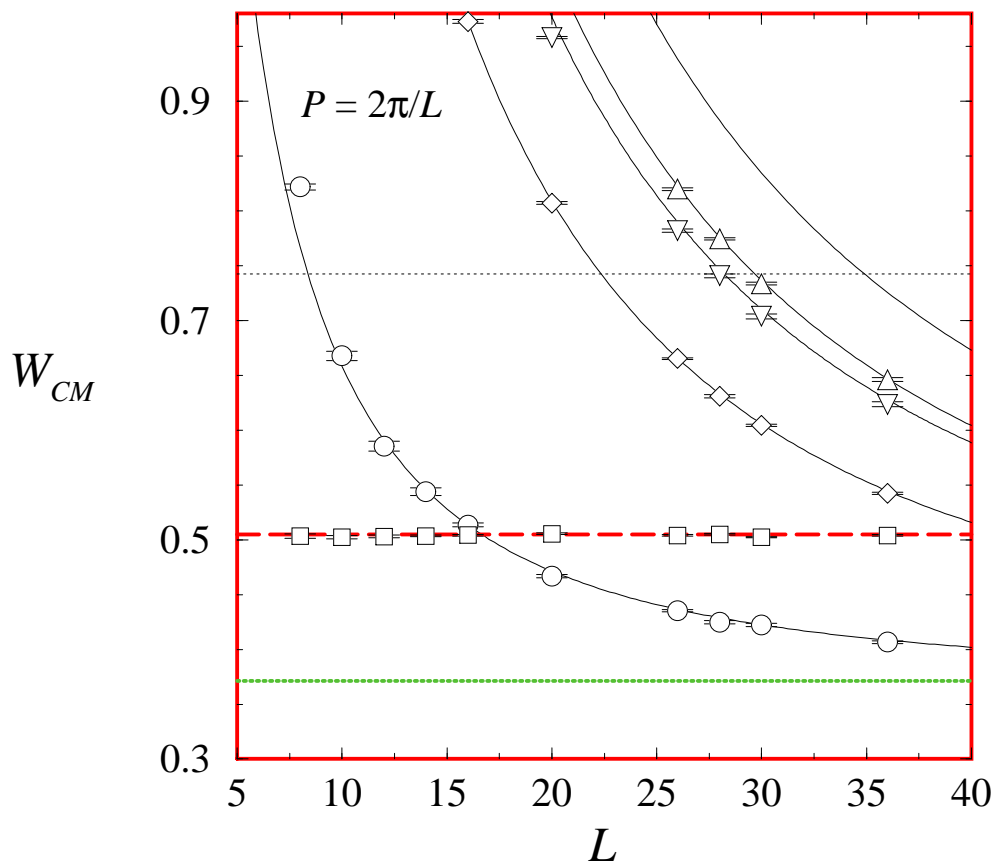
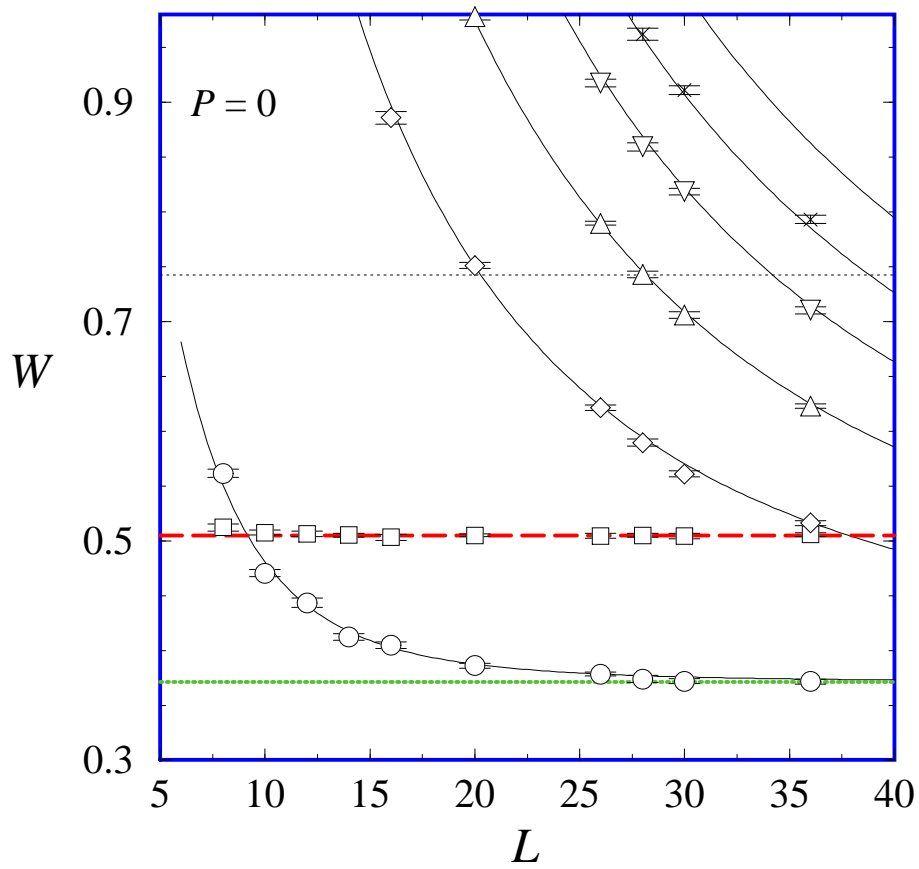
	A	B	C
$\kappa_\phi$	0.0742	0.07325	0.07075
$\kappa_\rho$	0.0708	0.0718	0.0665
$g$	0	0.008	0.021
# of lattice sizes	9	10	16
$m_\phi a$	0.1856(4)	0.1996(5)	0.3081(4)
$m_\rho a$	0.5049(5)	0.5306(13)	0.8206(11)
$\Gamma a$	0	0.0044(2)	0.0178(7)
$g_R a$	0	0.598(14)	1.49(3)
$\lambda_R$	28.1(1.1)	36.8(1.3)	48.3(2.0)

In case A  $g = 0$  and  $\rho$  is stable. In cases B and C  $g$  increases causing an increase in the resonance width  $\Gamma$ .  $g_R$  and  $\lambda_R$  are renormalized  $\phi\phi\rho$ - and  $\phi^4$  couplings, respectively.

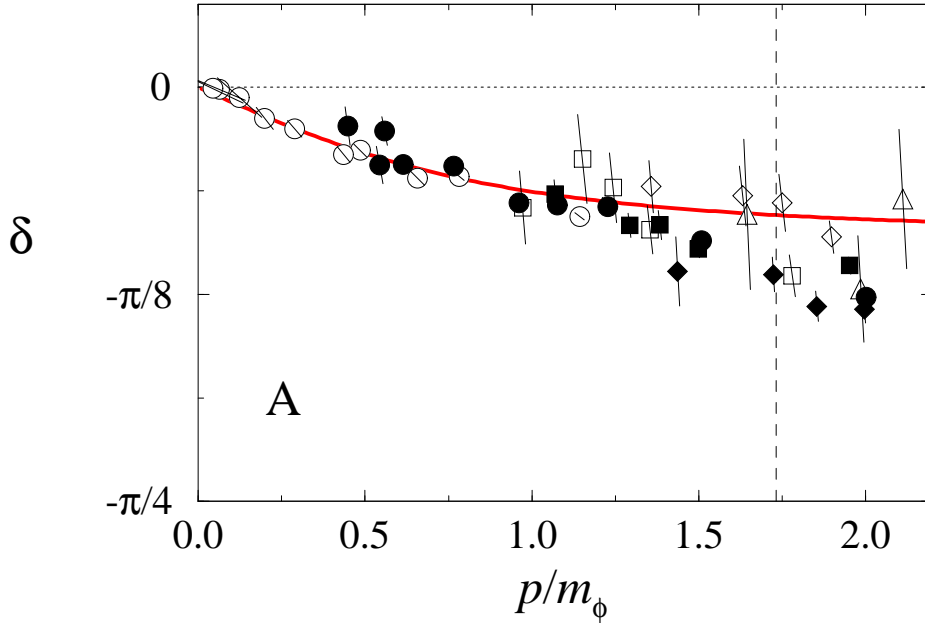
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<sup>1</sup>Gattringer and Lang used similar action to study resonance scattering in 1+1 dimensions [Nucl.Phys.B 391(1993)].

Case A energy spectrum.  $g = 0$ , and  $\rho$  is stable.



The phase shift is calculated from eqs. (1–4):

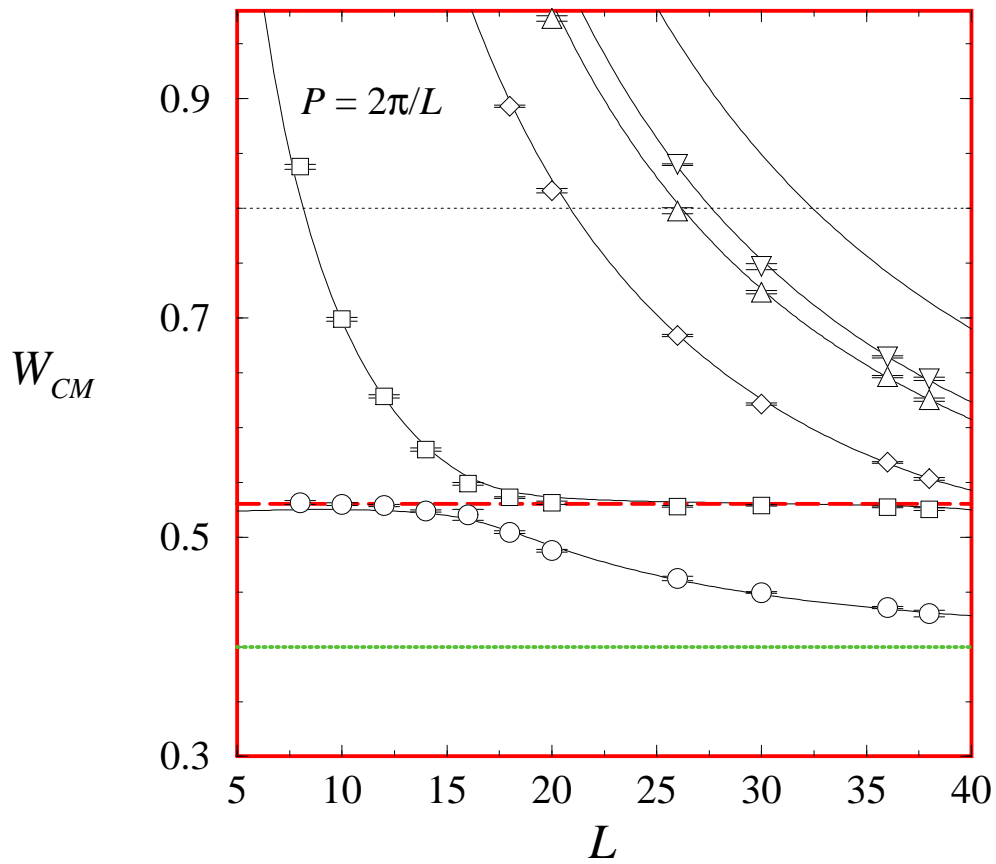
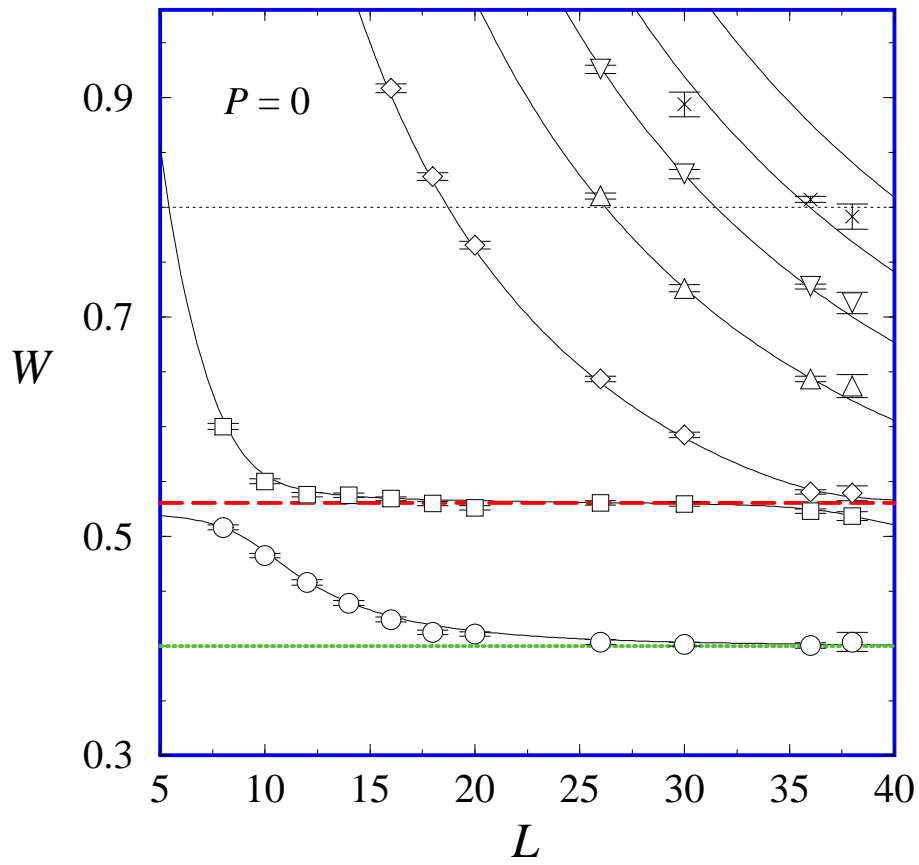


Open symbols correspond to  $\vec{P} = 0$  sector; filled symbols  $\vec{P} = 2\pi/L$ . Continuous line is a 1-parameter fit to the perturbative ansatz

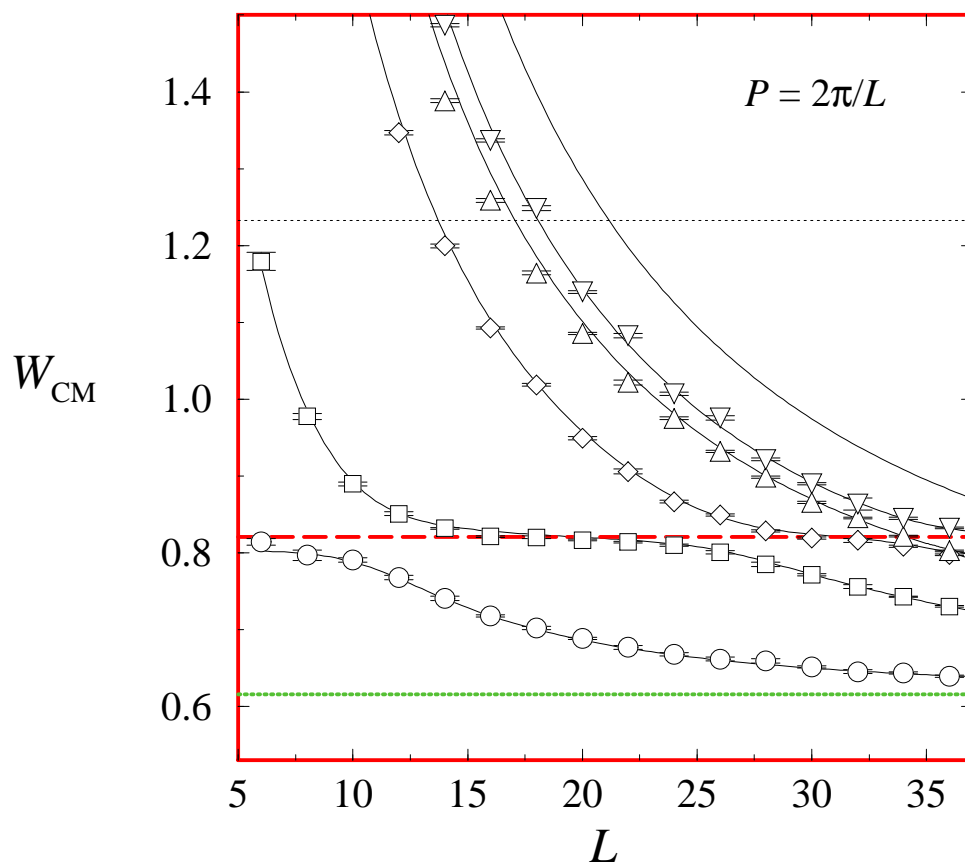
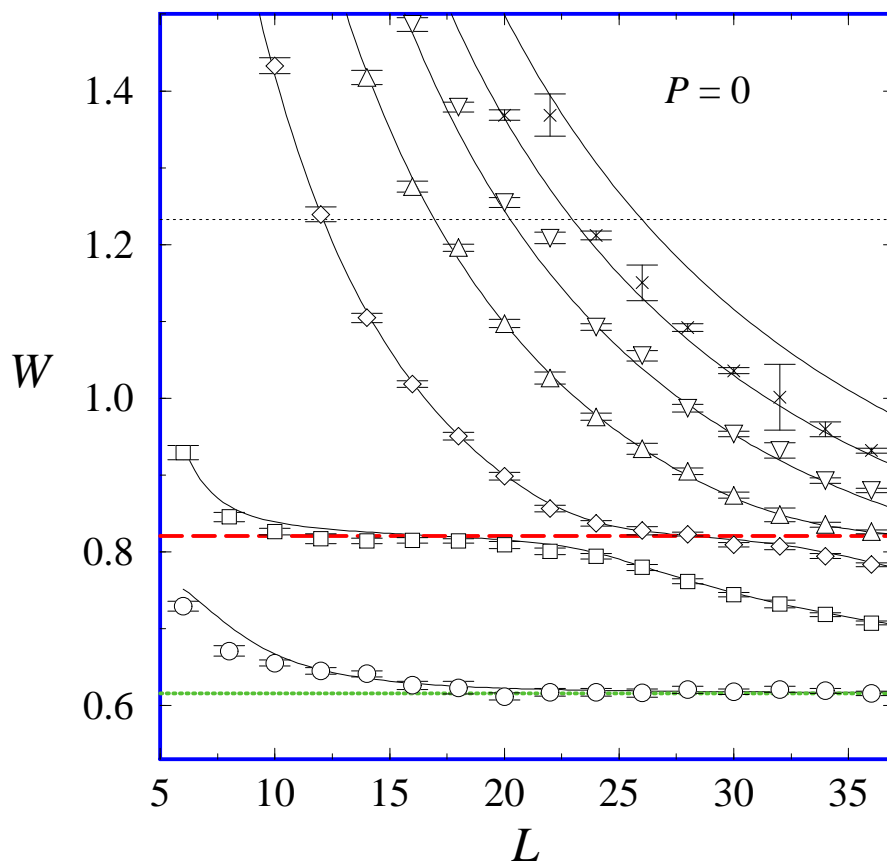
$$\delta_0^{\text{pert}} = -\lambda_R \frac{p}{16\pi W}$$

The lowest  $P = 0$  energy level increases sharply when  $L \rightarrow 0$ . This is due to the strong repulsive  $\phi^4$  interaction; this is also evidenced by the negative phase shift and scattering length.

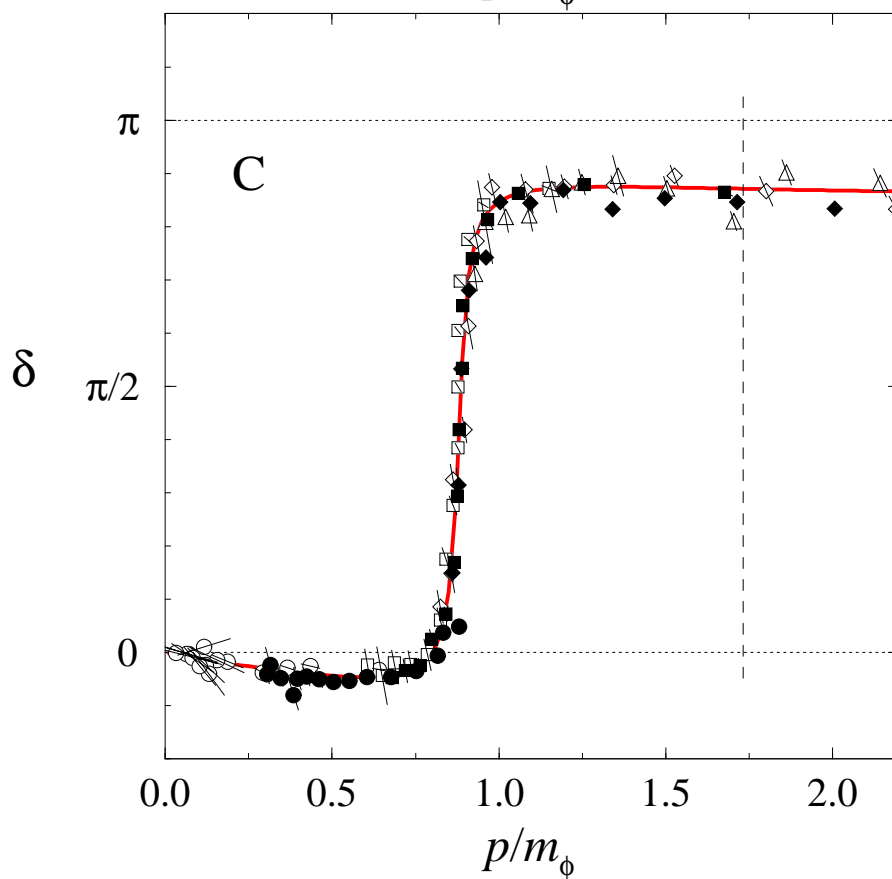
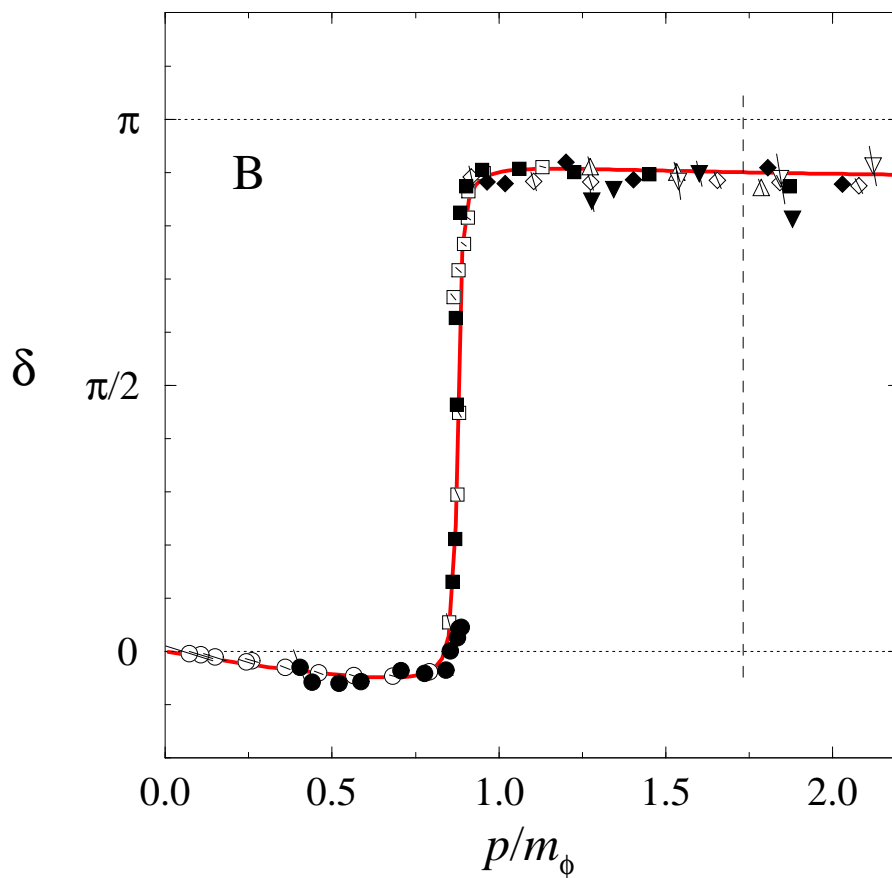
Case B:  $g = 0.008$



Case C:  $g = 0.021$



B and C (open symbols:  $\vec{P} = 0$ , filled symbols:  $\vec{P} = 2\pi/L$ ):



In cases B and C the resonance appears as a rapid increase of the phase shift at  $W_{\text{cm}} = m_\rho$ . The width of the resonance is proportional to the slope of  $\delta$  at  $\pi/2$ .

Continuous red lines: 3-parameter perturbative fits to  $\delta_0$ -data

$$\begin{aligned}\delta_0^{\text{pert}} &= \delta_r + \delta_s \\ \delta_r &= -\lambda_R \frac{p}{16\pi W} + \frac{g_R^2}{32\pi W p} \log \frac{4p^2 + m_\rho^2}{m_\rho^2} \\ \tan \delta_s &= -\frac{g_R^2}{16\pi W} \frac{p}{W^2 - m_\rho^2}.\end{aligned}$$

Using the ansatz with the fitted parameter we obtain the continuous lines in the energy level plots. Note that *all* the levels are fitted by the same parameter values.

The resonance width is calculated from

$$\Gamma_\rho = \frac{g_R^2}{32\pi m_\rho^2} \sqrt{m_\rho^2 - 4m_\phi^2}$$

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Also in cases B and C the repulsive  $\phi^4$  interaction increases the energy of the lowest  $\vec{P} = 0$  level when  $L$  small. This causes an avoided level crossing in case B at  $L = 8$ . However, at  $L \sim 8$  the resonance data is spoiled by the finite size effects. The first usable crossing occurs at  $L = 16$  in  $\vec{P} = 2\pi/L$  sector.

## Conclusions

- Lattice calculations are a viable tool for resolving the resonance scattering parameters.
- We have obtained very high quality Monte Carlo data from the resonance scattering in 3+1 dimensional lattice calculations.
- $\vec{P} = 0$  and  $\vec{P} = 2\pi/L$  sectors are completely compatible within statistical errors.
- Data agrees remarkably well with the perturbative ansatz.
- Level crossings occur in  $\vec{P} = 2\pi/L$  sector in much smaller lattice volumes than in  $\vec{P} = 0$  sector.
- The two sectors complement each other:
  - For a fixed volume, they give the phase shift function  $\delta_l(p)$  with different argument  $p$ .
  - Same configurations can be used to calculate both sectors, often at little extra cost in time.