3. Stochastic processes and fields

3.1 Langevin equation

Many equations describing evolution of physical, chemical, biological and social processes are written as mean-field equations for averages of quantities, which intrinsically are random processes to some extent. To take fluctuations around the averages into account, a straightforward way to proceed is to introduce a source of randomness directly in the mean-field equation. Then the quantities solved from the mean-field equations become *stochastic processes* depending on coordinate variables, i.e. *stochastic fields*.

■ **Brownian motion.** The paradigmatic example of this procedure is the Langevin equation for random walk, which describes the position $\boldsymbol{r}$ of a test particle subject to random force $\boldsymbol{\eta}$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{\eta}$$

Here, the random force is of zero mean and uncorrelated in time (white noise), i.e.

$$\langle \eta_i(t)\eta_j(t')\rangle = D\delta_{ij}\delta(t-t')\,.$$

Strictly speaking, this standard physical formulation is mathematically inconsistent, which gives rise to inevitable ambiguities in the case of multiplicative noise (when the noise term is multiplied by a function of the random position).

Consider, for instance, the average distance between two points of the path of the random walk, i.e. (d is the dimension of space)

$$\left\langle [\boldsymbol{r}(t)-\boldsymbol{r}(t')]^2 \right\rangle = Dd\int_0^t du \int_0^{t'} dv\delta(u-v) = Dd|t-t'|$$

– a hint to the property that the Brownian path, although continuous, is not differentiable anywhere

Increments of the Brownian path

$$\boldsymbol{W}(t) - \boldsymbol{W}(t_0) = \int_{t_0}^t dt\, \boldsymbol{\eta}(t)$$

constitute an extremely important random process, the *Wiener process*, whose conditional probability density is the Gaussian

$$p\left(\boldsymbol{W},t|\boldsymbol{W}_0,t_0\right) = \frac{1}{[4\pi(t-t_0)]^{d/2}}\, e^{-(\boldsymbol{W}-\boldsymbol{W}_0)^2/2(t-t_0)}\,.$$

In particular,

$$\langle \boldsymbol{W}(t)\rangle = \boldsymbol{W}_0\,, \quad \langle [W_i(t) - W_{0i}]\,[W_j(t) - W_{0j}]\rangle = (t - t_0)\delta_{ij}\,. \tag{3.1}$$

The Wiener process is the basis of the mathematically consistent definition of the Langevin equation (the stochastic differential equation, SDE).

■ **Critical dynamics.** In the Landau theory of phase transitions the dynamics of the order parameter φ near equilibrium are described by the kinetic equation [time-dependent Ginzburg-Landau (TDGL) equation]

$$\frac{\partial \varphi}{\partial t} = -\Gamma\left(-\nabla^2\varphi + a\varphi + \frac{\lambda}{6}\varphi^3\right)\,. \tag{3.2}$$

In linear response theory, dynamics of fluctuations near equilibrium [1] are often described by kinetic equations similar to (3.2), but with a random noise term added to the right-hand side.

In a more generic setup, standard models of critical dynamics are based on nonlinear Langevin equations

$$\frac{\partial \varphi}{\partial t} = -\Gamma\frac{\delta H}{\delta \varphi} + f := V(\varphi) + f\,, \tag{3.3}$$

where H is the effective equilibrium Hamiltonian. For the random source a suitable Gaussian distribution is assumed in which the correlation function is determined through the connection to the static equilibrium (fluctuation-dissipation theorem). For instance, model A for the non-conserved order parameter [2] is described by the SDE obtained from 3.2 by the addition of a white-noise field to the right-hand side.

■ **Diffusion-limited reactions.** In reaction kinetics and population dynamics the simplest kinetic description of the dynamics of the average particle numbers is given by the *rate equation*. The rate equation is a deterministic differential equation for average particle numbers in a homogeneous system, therefore it does not take into account boundary conditions, spatial inhomogeneities and randomness in the individual reaction events. Spatial dependence is often accounted for by a diffusion term, which gives rise to models of *diffusion-limited reactions* (DLR).

As a simple example, consider the coagulation reaction $A + A \to A$. The diffusion-limited rate equation for the concentration φ of the compound A is

$$\frac{\partial \varphi}{\partial t} = D\nabla^2\varphi - k\varphi^2\,,$$

where k is the *rate constant*.

The most straightforward way to take into account various effects of randomness is to add a random source and sink term to the rate equation:

$$\frac{\partial \varphi}{\partial t} = D\nabla^2\varphi - k\varphi^2 + f\,. \tag{3.4}$$

This is a nonlinear Langevin equation for the field φ. Physically, in the case of concentration $\varphi \geq 0$.

There is an important difference between the reaction models and the critical dynamics: in the latter, deviations of the fluctuating order parameter from the (usually zero) mean may physically be of any sign (or direction). In particular, deviations from the equilibrium value are always allowed. In the reaction there is often an absorbing steady state, which does not permit fluctuations therefrom: once the system arrives at the absorbing state, it stays there forever. In particular, if the empty state is an absorbing state of the reaction, the the random source should be introduced multiplied by a factor vanishing in the limit $\varphi \to 0$ to prevent the system returning from the absorbing state by the noise. The simplest choice yields

$$\frac{\partial \varphi}{\partial t} = D\nabla^2 \varphi - k\varphi^2 + f\varphi$$

instead of (3.4). This is an equation with a *multiplicative noise*.

■ **Multiplicative noise.** Consider the Langevin equation with the multiplicative noise of generic form

$$\frac{\partial \varphi}{\partial t} = V(\varphi) + fb(\varphi) := -K\varphi + U(\varphi) + fb(\varphi)\,, \tag{3.5}$$

where f is (usually) a Gaussian random field with zero mean and the white-in-time correlation function

$$\langle f(t,\boldsymbol{x}) f(t',\boldsymbol{x}')\rangle = \overline{D}(x-x') = \delta(t-t')D(\boldsymbol{x}-\boldsymbol{x}')\,, \tag{3.6}$$

where the shorthand notation $x = (t,\boldsymbol{x})$ has been used. In (3.5), $b(\varphi)$ is a functional of φ and $U(\varphi)$ is a nonlinear functional of φ. Both functionals are time-local, i.e. depend only on the current times instant of the SDE.

The Langevin equation with white-in-time noise f is mathematically inconsistent, because the time integral of the noise $\int f dt$ is a Wiener process which not differentiable anywhere as a function of time.

This problem may be approached by starting with the set of correlation functions consisting of a δ sequence in time, i.e.

$$\langle f(t,\boldsymbol{x}) f(t',\boldsymbol{x}')\rangle = \overline{D}(t,\boldsymbol{x};t',\boldsymbol{x}') \xrightarrow[t'\to t]{} \delta(t-t')D(\boldsymbol{x},\boldsymbol{x}') \tag{3.7}$$

and passing to the white-noise limit at a later stage. From the mathematical point of view, this treatment gives rise to the solution of the stochastic differential equation (3.5) in the Stratonovich sense [3]. Physically, this is the most natural way to approach the white-noise case.

■ **Iterative solution of the SDE.** Consider the stochastic differential equation with a multiplicative white-in-time noise W of the form

$$d\varphi = -K\varphi dt + b(\varphi)dW\,. \tag{3.8}$$

Here, the deterministic part is chosen linear in φ for simplicity whereas the coefficient function $b(\varphi)$ of the stochastic part is assumed to be a polynomial function of φ. In case of multiplicative white noise the Langevin form of the SDE (3.5) is mathematically inconsistent, therefore the equation has been written in the integral form, in which dW is the increment of a Wiener process for each value of the spatial coordinate $\boldsymbol{x}$.

Let us construct a solution for the SDE (3.8) by standard iterations. This procedure looks much simpler, if the SDE is first Fourier transformed with respect to the spatial variables and assumes the form (the same symbol is used for the original field and its Fourier transform)

$$\varphi(t,\boldsymbol{k}) = \varphi(0,\boldsymbol{k}) - K\int\limits_0^t \varphi(t',\boldsymbol{k})dt' + \int\limits_0^t\int \frac{d\boldsymbol{q}}{(2\pi)^d}\, b(\varphi,\boldsymbol{q})dW(t',\boldsymbol{k}-\boldsymbol{q})\,, \qquad (3.9)$$

where K is now a constant and $b(\varphi,\boldsymbol{q})$ is the spatial Fourier transform of the coefficient function $b(\varphi)$ of (3.8) and $dW(t,\boldsymbol{k})$ is the that of the increment of the Wiener process. In (3.9) the function $b(\varphi)$ is expressed as a functional of the field φ in the Fourier representation. Further, it is convenient to construct the iterative solution for the Laplace transform $\phi(s,\boldsymbol{k})$ with respect to time of the field $\varphi(t,\boldsymbol{k})$. The first two steps of the iteration process yield

$$\phi_0(s,\boldsymbol{k}) = \frac{\varphi(0,\boldsymbol{k})}{s+K}\,,$$
$$\phi_1(s,\boldsymbol{k}) = \frac{1}{s+K}\int\limits_0^\infty e^{-st}\int \frac{d\boldsymbol{q}}{(2\pi)^d}\, b(\phi_0,\boldsymbol{q})dW(t,\boldsymbol{k}-\boldsymbol{q})\,.$$

Here, $\varphi(0,\boldsymbol{k})$ is the Fourier transform of the initial value of the problem.

■ **Graphical representation. Multiplicative linear white noise.** The iterative solution of (3.9) is readily expressed in terms of a tree-graph expansion, when the coefficient function $b(\varphi)$ is a polynomial function of φ. Consider the simplest choice of linear function, i.e. the SDE of the *multiplicative linear white-noise* process

$$\frac{\partial\varphi}{\partial t} = -K\varphi + f\varphi\,, \qquad (3.10)$$

where K is a time-independent operator acting on the field φ (e.g. $K = -\nabla^2 + a$, $a > 0$) and f a Gaussian random field with zero mean. Difficulties in the interpretation of the equation (3.10) arise, when the correlation function of the field f is local in time, i.e. when

$$\langle f(t,\boldsymbol{x})f(t',\boldsymbol{x}')\rangle = \delta(t-t')D(\boldsymbol{x},\boldsymbol{x}')\,. \qquad (3.11)$$

Recall that for the analysis of the white noise limit we have used the set of correlation functions consisting of a δ sequence in time (3.7)

$$\langle f(t,\boldsymbol{x})f(t',\boldsymbol{x}')\rangle = \overline{D}(t,\boldsymbol{x};t',\boldsymbol{x}') \xrightarrow[t'\to t]{} \delta(t-t')D(\boldsymbol{x},\boldsymbol{x}')\,,$$

which gives rise to the solution of the SDE (3.10) in the Stratonovich sense.

The iterative solution of the SDE (3.10) may expressed as the series

$$\varphi = \Delta\chi + \Delta f \Delta\chi + \Delta f \Delta f \Delta\chi + \ldots \tag{3.12}$$

where χ is the initial condition of the solution

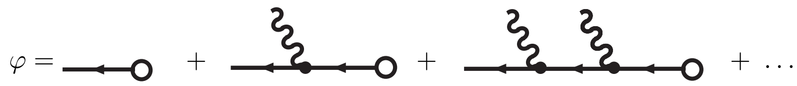
$$\Delta\chi = \int d\boldsymbol{x}' \Delta(t, \boldsymbol{x} - \boldsymbol{x}')\chi(\boldsymbol{x}')$$

of the homogeneous equation

$$\left(\frac{\partial}{\partial t} + K\right)\Delta\chi = 0\,, \tag{3.13}$$

where Δ is the (retarded) Green function of the same equation.

In (3.12) a shorthand notation has been used in which all time and space convolution integrals in the nonlinear terms are implied. Graphically, the solution is a sum of chains of identically oriented lines corresponding to retarded propagators



(3.14)

where the circle stands for the initial condition χ of the homogeneous equation (3.13), the wavy line corresponds to the random field f and the full dot represents the vertex factor brought about by the last term of the Langevin equation (3.10). Formally, the solution (3.14) is the tree-graph representation of the solution of the stochastic differential equation (3.10) with the additive source field $\delta(t)\chi(\boldsymbol{x})$.

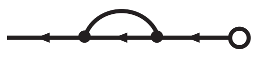
The perturbative solution of the SDE (3.10) is given by Wick's theorem for the Gaussian distribution of f, which graphically amounts to replacing any pair of f by the correlation function $\overline{D}$ depicted by an unoriented line in all possible ways and – in case of zero mean – discarding all graphs with an odd number of wavy lines. For instance, the graphical expression (3.14) gives rise to the representation

(3.15)

Since the free-field differential operator in (3.10) is of first order in time, the physically interesting Green function is the retarded propagator. In the limit of white-in-time correlations this leads to an enormous truncation of the averaged iterative series (3.12), because it brings about temporal δ

functions contracting the ends of chains of the retarded propagators. Any graph containing at least one such closed loop of at least two propagators vanishes. Only those terms, in which the correlation function is multiplied by a single retarded propagator do not vanish automatically.

For instance, the one-loop graph in (3.15) in case of white-in-time noise gives rise to an ambiguity, which is directly related to that in the interpretation of the stochastic differential equation (3.10). A straightforward substitution of the white-noise correlation function in this graph gives rise to the expression



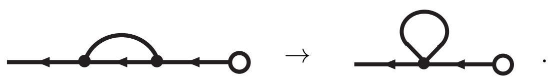
$$= \int dt_1 \int d\boldsymbol{x}_1 \int d\boldsymbol{x}_2 \int d\boldsymbol{x}_3 \, \Delta(t - t_1, \boldsymbol{x} - \boldsymbol{x}_1) \times \Delta(0, \boldsymbol{x}_1 - \boldsymbol{x}_2) D(\boldsymbol{x}_1, \boldsymbol{x}_2) \Delta(t_1, \boldsymbol{x}_2 - \boldsymbol{x}_3) \chi(\boldsymbol{x}_3) \,, \quad (3.16)$$

where the value of the propagator at coinciding time arguments $\Delta(0, \boldsymbol{x}_1 - \boldsymbol{x}_2) = \theta(0)\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ is ambiguous.

With the use of the δ-sequence of correlation functions (3.12) this ambiguity is readily resolved and gives rise to the expression

$$= \frac{1}{2} \int dt_1 \int d\boldsymbol{x}_1 \int d\boldsymbol{x}_3 \, \Delta(t - t_1, \boldsymbol{x} - \boldsymbol{x}_1) D(\boldsymbol{x}_1, \boldsymbol{x}_1) \Delta(t_1, \boldsymbol{x}_1 - \boldsymbol{x}_3) \chi(\boldsymbol{x}_3) \,, \quad (3.17)$$

with the coefficient $\frac{1}{2}$ in front of the spatial δ function. As noted before, this procedure corresponds to the interpretation of the SDE (3.10) in the Stratonovich sense. Formally, this result may be obtained by amending the definition of the propagator according to the rule $\Delta(0, \boldsymbol{x} - \boldsymbol{x}') = \frac{1}{2}\delta(\boldsymbol{x} - \boldsymbol{x}')$. Within this choice the graphical expression in (3.17) appears excessive, because it hints to twice the number of integrations than actually is carried out. However, the white-noise limit may graphically be depicted as replacement of the one-loop graph with the noise correlation function by a new vertex factor with the coefficient $\frac{1}{2}D(0)$:

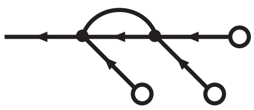


A more convenient description may be obtained, if we adopt the convention $\Delta(0) = 0$, which corresponds to the the interpretation of the SDE (3.10) in the Ito sense. In this case the graph (3.17) vanishes in the white-noise limit emphasizing the fact that the choice of the interpretation of the SDE affects its solution.

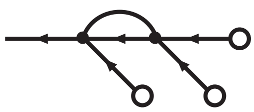
■ **Graphical representation. Multiplicative quadratic white noise.** A similar prescription holds in the direct construction of iterative solutions of stochastic differential equations, when the random source field is multiplied by a polynomial function(al) of the field φ as well. Consider the SDE with the white noise multiplied by the field squared

$$\frac{\partial \varphi}{\partial t} = -K\varphi + \frac{1}{2} f \varphi^2 \tag{3.18}$$

as an example. Instead of (3.15) the first terms of the perturbation expansion of the solution are [assume the δ sequence (3.12) of the correlation functions of f for the time being]

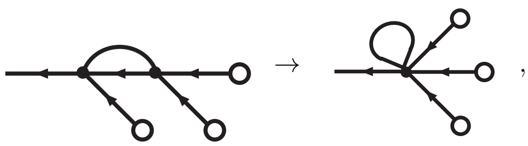
$\langle \varphi \rangle =$  $+$  $+ \ldots$ (3.19)

The analytic expression in the white-noise limit assumes the form



$$= \frac{1}{2} \int dt_1 \int d\boldsymbol{x}_1 \int d\boldsymbol{x}_2 \int d\boldsymbol{x}_3 \int d\boldsymbol{x}_4 \, \Delta(t-t_1, \boldsymbol{x}-\boldsymbol{x}_1)$$
$$\times \, D(\boldsymbol{x}_1, \boldsymbol{x}_1) \Delta(t_1, \boldsymbol{x}_1 - \boldsymbol{x}_2) \chi(\boldsymbol{x}_2) \Delta(t_1, \boldsymbol{x}_1 - \boldsymbol{x}_3) \chi(\boldsymbol{x}_3) \Delta(t_1, \boldsymbol{x}_1 - \boldsymbol{x}_4) \chi(\boldsymbol{x}_4) \,, \tag{3.20}$$

which corresponds to the Stratonovich SDE. Graphically, this operation again reduces the one-loop white-noise subgraph to a dot:



whereas in the case of the Ito SDE this term vanishes in the white-noise limit.

■ **Resolution rules.** As the previous examples demonstrate, the ambiguity in the solution of (3.8) shows, when averages of functions of φ are calculated. The rather clumsy handicraft rule inferred above may be streamlined with the use the properties of the Ito and Stratonovich stochastic integrals.

I am not going to dwell on the mathematics here, however. The practical prescriptions are the same as in the previous examples.

Let us note first that the combinatorial rule for the Wiener process is given by the usual Wick theorem. More formally than in the examples, the result of calculation of time integrals of pairwise products of the Wiener process in case of Ito stochastic integral is given by the correlation formula [3]

$$\left\langle \int_{t_0}^{t} G(u)dW(u) \int_{t_0}^{t} H(v)dW(v) \right\rangle = \int_{t_0}^{t} \langle G(u)H(u)\rangle du \,, \tag{3.21}$$

where $G(t)$ and $H(t)$ are continuous nonanticipating functions and the brackets denote average over the probability distribution of the Wiener process. It should be noted that all functions of time emerging in the iterative solution of the SDE (3.8) are nonanticipating. The corresponding formula for the Stratonovich integral will not be quoted here, because the following discussion is based on simple practical rules for evaluation of correlation functions of the Wiener field.

The practical rule of calculation may be stated as follows [3]: rewrite the increment of the Wiener process as

$$dW(t) \to \xi(t)dt \,,$$

where $\xi(t)$ is δ-correlated in time, i.e.

$$\langle \xi(t)\xi(t')\rangle = \delta(t-t') \,,$$

and calculate the averages in terms of ξ. With the use of this rule, integrals appear in which the δ function of correlations of ξ is integrated in such a way that one of the limits of integration coincides with one of the arguments of the δ function. With the use of Ito stochastic integral, ambiguities arising in these case are resolved as

$$\begin{aligned} \int_{t_1}^{t_2} f(t)\delta(t-t_1) &= f(t_1) \,, \\ \int_{t_1}^{t_2} f(t)\delta(t-t_2) &= 0 \,. \end{aligned} \tag{3.22}$$

In case of the Stratonovich stochastic integral these rules are replaced by

$$\begin{aligned} \int_{t_1}^{t_2} f(t)\delta(t-t_1) &= \frac{1}{2} f(t_1) \,, \\ \int_{t_1}^{t_2} f(t)\delta(t-t_2) &= \frac{1}{2} f(t_2) \,. \end{aligned} \tag{3.23}$$

A little reflection shows that in the tree-graph iterative solution these situations arise solely due to the presence of the temporal step function in the propagator.

3.2 Fokker-Planck equation

Instead of the mathematically problematic, although physically transparent, Langevin equation the stochastic problems at hand may be equivalently stated in terms of the Fokker-Planck equation (FPE), which is an equation for both the conditional probability density $p(\varphi, t|\varphi_0, t_0)$ and the probability density $p(\varphi, t)$. The simple way to demonstrate this equivalence uses rules of Ito calculus [3], however, and I am not going to dwell on this issue here. Other seemingly simple methods use – at least implicitly – functional integrals, which are still mathematically ill-defined, let alone the Gaussian integral. Therefore, only the correspondence between the quantities specifying the stochastic problem in both approaches will be quoted here. The main advantage of the Fokker-Planck equation is that the equation itself is completely well-defined partial differential (or functional-differential for field variables) equation. The ambiguity of the Langevin problem shows in that the FPE is different for different interpretations of the SDE.

■ **Fokker-Planck equation and Langevin equation.** For simplicity of notation, consider zero-dimensional field-theory, i.e. the stochastic process defined by the SDE (3.5) with φ considered a function of time only:

$$\frac{\partial\varphi}{\partial t} = -K\varphi + U(\varphi) + fb(\varphi)\,, \tag{3.24}$$

with the correlation function of the noise

$$\langle f(t)f(t')\rangle = \overline{D}(t-t') = \delta(t-t')D\,. \tag{3.25}$$

The corresponding Fokker-Planck equation for the conditional probability density $p(\varphi, t|\varphi_0, t_0)$ in the case of the Ito equation is

$$\frac{\partial}{\partial t}p(\varphi, t|\varphi_0, t_0) = -\frac{\partial}{\partial\varphi}\left\{[-K\varphi + U(\varphi)]\,p(\varphi, t|\varphi_0, t_0)\right\} + \frac{1}{2}\frac{\partial^2}{\partial\varphi^2}\left[b(\varphi)Db(\varphi)p(\varphi, t|\varphi_0, t_0)\right]\,. \tag{3.26}$$

The conditional probability density $p(\varphi, t|\varphi_0, t_0)$ is the fundamental solution of the FPE (3.26), i.e. the initial condition is

$$p(\varphi, t_0|\varphi_0, t_0) = \delta(\varphi - \varphi_0)\,.$$

If the SDE (3.24) is interpreted in the Stratonovich sense, the FPE is

$$\frac{\partial}{\partial t}p\left(\varphi,t|\varphi_0,t_0\right)=-\frac{\partial}{\partial\varphi}\left\{\left[-K\varphi+U(\varphi)\right]p\left(\varphi,t|\varphi_0,t_0\right)\right\}$$
$$+\frac{1}{2}\frac{\partial}{\partial\varphi}\left\{b(\varphi)\frac{\partial}{\partial\varphi}\left[Db(\varphi)p\left(\varphi,t|\varphi_0,t_0\right)\right]\right\}. \quad (3.27)$$

Contractions are not obvious, when the variable φ has several components. For instance, the Fokker-Planck equation in the Ito form becomes

$$\frac{\partial}{\partial t}p\left(\varphi,t|\varphi_0,t_0\right)=-\frac{\partial}{\partial\varphi_i}\left\{\left[-K_{ij}\varphi_j+U_i(\varphi)\right]p\left(\varphi,t|\varphi_0,t_0\right)\right\}$$
$$+\frac{1}{2}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j}\left[b_{ik}(\varphi)D_{kl}b_{jl}(\varphi)p\left(\varphi,t|\varphi_0,t_0\right)\right]$$

in that case.

■ **The Fokker-Planck equation and the Schrödinger equation.** The important point for the subsequent presentation is that the FPE may be written in the form of a Schrödinger equation with imaginary time

$$\frac{\partial}{\partial t}p=\widehat{L}p\,, \quad (3.28)$$

where $\widehat{L}$ is the *Liouville operator*. The usual Schrödinger equation assumes this form after the substitution $t\to -i\hbar t$:

$$i\hbar\frac{\partial}{\partial t}\Psi=\widehat{H}\Psi\ \longrightarrow\ -\frac{\partial}{\partial t}\Psi=\widehat{H}\Psi\,,$$

which is actually used in quantum statistical physics. The formal solution of the Liouville equation (3.28) may be written as

$$p(t)=e^{\widehat{L}(t-t_0)}p(t_0)\,.$$

in analogy with the representation of the quantum-mechanical evolution in the form

$$\Psi(t)=e^{-i\widehat{H}(t-t_0)/\hbar}\Psi(t_0)\,.$$

The Liouville operator generating the time evolution is not Hermitian for the FPE, but this is unimportant for the subsequent construction of the evolution operator in the quantum-mechanical fashion. We shall see shortly how this analogy with quantum mechanics allows to use the machinery of quantum field theory for construction of the solution of the Fokker-Planck equation.

3.3 Master equation

Markov processes described in terms of the Fokker-Planck equation have continuous sample paths. Not all interesting stochastic processes belong to this category. A wide class of such processes describe changes in occupation numbers (e.g. individuals of some population, molecules in chemical reaction) which cannot be naturally described by continuous paths. This kind of processes are described by *master equations* – a special case of (differential) Kolmogorov equations [3].

■ **The master equation.** The generic form of a master equation written for the conditional probability density $p\,(\varphi, t|\varphi_0, t_0)$ of a Markov process is

$$\frac{\partial}{\partial t} p\,(\varphi, t|\varphi_0, t_0) = \int d\chi \left[W(\varphi|\chi, t) p\,(\chi, t|\varphi_0, t_0) - W(\chi|\varphi, t) p\,(\varphi, t|\varphi_0, t_0) \right] , \quad (3.29)$$

where $W(\varphi|\chi, t)$ is the *transition probability* per unit time, whose formal definition from the differential Kolmogorov equation is (for all $\varepsilon > 0$)

$$W(\varphi|\chi, t) = \lim_{\Delta t \to 0} \frac{p\,(\varphi, t + \Delta t|\chi, t)}{\Delta t} ,$$

uniformly in φ, χ and t for all $|\varphi - \chi| \geq \varepsilon$.

We shall be using the master equation for discrete variables, in this case the discontinuous character of the paths of the *jump processes* described by the master equation is especially transparent:

$$\frac{\partial}{\partial t} p\,(n, t|m, t_0) = \sum_l \left[W(n|l, t) p\,(l, t|m, t_0) - W(l|n, t) p\,(n, t|m, t_0) \right] . \quad (3.30)$$

This set of master equations shall also be cast in the form of an evolution equation of the type of Schrödinger equation, but the representation is not as straightforward as in the case of the Fokker-Planck equation.